A new Bayesian Backtesting framework: Application to Non-Linear Mortality modeling

Melvern Leung

Supervisors: A.Prof. Athanasios Pantelous A.Prof Catherine Forbes

Econometrics and Business Statistics (Actuarial)

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Overview

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- Produce a new method of backtesting specifically tiered to Longevity risk management.
- Cast the famous Lee-Carter model and Cairns-Blake-Dowd Model in a Bayesian (non-linear) state-space form under the Poisson/Binomial Error structure.
- Application of the backtesting approach to compare whether the Poisson/Binomial Error structure provides a better tool for mortality modeling than the linear state-space form.

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- Life annuity, deferred annuity, and variable annuity are all products that hinges on longevity risk.
- Recently longevity risk has been exacerbated by consistent mis-estimation of predicted mortality rates and life expectancy
- Bayesian Linear mortality models have been used extensively in literature as a means to provide better fit but there is an issue!
- Can we create a framework which determines whether a model is "good enough" to capture longevity risk? More specifically, managing longevity risk?

The issue with Null-Hypothesis Significance-Testing (NHST)

- The question whether we want to answer

$$P(\text{Data}|\text{H}_0) \quad \text{or} \quad P(\text{H}_0|\text{Data}). \tag{1}$$

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- Indirect inference on the alternative as the p-value is conditional on the Null.
- Under a Bayesian Hypothesis Testing, we are aiming to find $\text{Pr}(\text{H}_0|\text{Data})$
- Transparency under Bayesian Hypothesis Testing

The Bayesian Backtesting Framework

- Our new backtesting framework tries to combine both the area of Bayesian Hypothesis testing, and the Kupiec's Unconditional Coverage test. (Bayesian is used as an alternative means to Frequentist testing).

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- First approach is using the Bayes Factor (B_{01})
- Second approach is using the Bayesian Likelihood Ratio test. ($BLRT_{01}$)
- Test for robustness using varying hyper-parameters.

Bayesian Mortality Model

- Introduce the two most commonly used mortality models, Lee-Carter (Lee and Carter, 1992), and the Cairns Blake Dowd (CBD) model (Cairns et al., 2006).
- Casting Lee-Carter and CBD model in State-Space form.
- 2 Estimation methods, Bayesian Non-linear State-Space model, and the Bayesian linear State-Space method under Gaussian error (see Fung et al. (2015); Leung et al. (2018); Pedroza (2006))

Notations

- age vector $\boldsymbol{x} := (x_1, ..., x_n)$,
- time vector $\boldsymbol{t} := (t_1, ..., t_T)$
- constant force of mortality assumption and we denote this by $m_{x,t}$.

$$\mu_{x+s,t+s} = \mu_{x,t} = m_{x,t} \quad \text{for } 0 \leq s < 1 \text{ and } x \in \mathbb{N}.$$

- $D_{x,t}$= No. of deaths, $E_{x,t}$ = central exposure

The Problem

Let the crude mortality rate be denoted as $\tilde{m}_{x,t} = \frac{D_{x,t}}{E_{x,t}}$. Let, $y_{x,t} := \log(\tilde{m}_{x,t})$, the Lee-Carter model assumes that

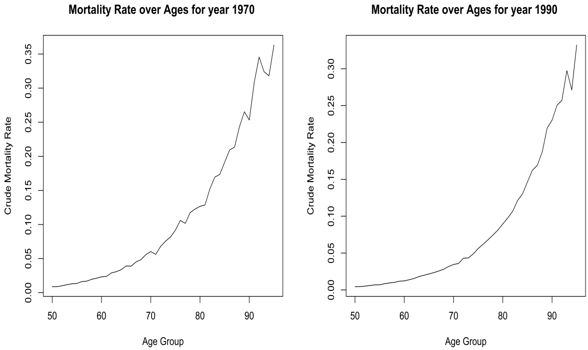
$$\boldsymbol{y_t} = \boldsymbol{\alpha} + \boldsymbol{\beta}\kappa_t + \boldsymbol{\varepsilon_t}, \quad \varepsilon_{\mathbf{t}} \sim N(\mathbf{0}, \mathbb{I}_n \sigma_\varepsilon^2)$$

$$\kappa_t = \kappa_{t-1} + \mu + \omega_t, \quad \omega_t \sim N(0, \sigma_\omega^2)$$

Let the probability of death be denoted as $\tilde{q}_{x,t} = -\log(1 - \tilde{m}_{x,t})$. CBD Model propose to model the probability of death as,

$$\log\left(\frac{\tilde{q}_{x,t}}{1-\tilde{q}_{x,t}}\right) = \kappa_{1,t} + \kappa_{2,t}(x - \bar{x}) + \varepsilon_{x,t}, \quad \varepsilon_{x,t} \sim N(0, \sigma_\varepsilon^2)$$

$$\begin{bmatrix}\kappa_{1,t}\\ \kappa_{2,t}\end{bmatrix} = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \begin{bmatrix}\kappa_{1,t-1}\\ \kappa_{2,t-1}\end{bmatrix} + \begin{bmatrix}\omega_{1,t}\\ \omega_{2,t}\end{bmatrix}, \quad \begin{bmatrix}\omega_{1,t}\\ \omega_{2,t}\end{bmatrix} \sim N(\mathbf{0}, \Sigma),$$


Mortality Rate over Ages for year 1970
Crude Mortality Rate
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
50
60
70
80
90
Age Group
Mortality Rate over Ages for year 1990
Crude Mortality Rate
0.00
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Dynamic Linear Model

Natural Exponential Family: $\log(\Pr(\boldsymbol{y}|\eta)) = \boldsymbol{y}^T\eta - b(\eta) + c(\boldsymbol{y})$
Assuming that death rates $D_{x,t}$ follows a Poisson distribution

$$\Pr(\mathrm{D}_{x,t} = \mathrm{d}_{x,t}|\mathrm{E}_{x,t}, \mathrm{m}_{x,t}) = \frac{e^{-E_{x,t}m_{x,t}}(E_{x,t}m_{x,t})^{\mathrm{d}_{x,t}}}{\mathrm{d}_{x,t}!},$$

$$\log(\Pr(\mathrm{D}_{x,t} = \mathrm{d}_{x,t}|\mathrm{E}_{x,t}, \mathrm{m}_{x,t})) = \mathrm{d}_{x,t}\log(E_{x,t}m_{x,t}) - E_{x,t}m_{x,t} - \log(\mathrm{d}_{x,t}!)$$

If death rates $D_{x,t}$ follows a Binomial Distribution then,

$$\Pr(\mathrm{D}_{x,t} = \mathrm{d}_{x,t}|\mathrm{E}_{x,t}, \mathrm{q}_{x,t}) = \binom{\mathrm{E}_{x,t}}{\mathrm{d}_{x,t}} \mathrm{q}_{x,t}^{\mathrm{d}_{x,t}}(1-\mathrm{q}_{x,t})^{\mathrm{E}_{x,t}-\mathrm{d}_{x,t}}$$

$$\log(\Pr(\mathrm{D}_{x,t=\mathrm{d}_{x,t}}|\mathrm{E}_{x,t}, \mathrm{q}_{x,t})) = \mathrm{d}_{x,t}\log\left(\frac{\mathrm{q}_{x,t}}{1-\mathrm{q}_{x,t}}\right) + \mathrm{E}_{x,t}\log\left(1-\mathrm{q}_{x,t}\right) + \log\binom{\mathrm{E}_{x,t}}{\mathrm{d}_{x,t}}$$

Non-Linear Mortality Model

Under Lee-Carter Poisson Model

$$
\begin{aligned}
(\mathrm{D}_{x,t}|\mathrm{E}_{x,t}\mathrm{m}_{x,t}) &\sim \frac{e^{-E_{x,t}m_{x,t}}(E_{x,t}m_{x,t})^{\mathrm{d}_{x,t}}}{\mathrm{d}_{x,t}!} \\
\log(\mathrm{m}_{x,t}) &= (\alpha_x + \beta_x\kappa_t)\,, \\
\kappa_t &= \kappa_{t-1} + \mu + \omega_t,
\end{aligned}
$$

Under CBD Binomial Model,

$$
\begin{aligned}
(\mathrm{D}_{x,t}|\mathrm{E}_{x,t}, \mathrm{q}_{x,t}) &\sim \binom{\mathrm{E}_{x,t}}{\mathrm{d}_{x,t}} \mathrm{q}_{x,t}^{\mathrm{d}_{x,t}}(1-\mathrm{q}_{x,t})^{\mathrm{E}_{x,t}-\mathrm{d}_{x,t}} \\
\log\left(\frac{q_{x,t}}{1-q_{x,t}}\right) &= \kappa_{1,t} + \kappa_{2,t}(x-\bar{x}), \\
\begin{bmatrix}\kappa_{1,t}\\ \kappa_{2,t}\end{bmatrix} &= \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} + \begin{bmatrix}\kappa_{1,t-1}\\ \kappa_{2,t-1}\end{bmatrix} + \begin{bmatrix}\omega_{1,t}\\ \omega_{2,t}\end{bmatrix}, \quad \begin{bmatrix}\omega_{1,t}\\ \omega_{2,t}\end{bmatrix} \sim N(\mathbf{0}, \Sigma).
\end{aligned}
$$

Bayesian Non-Linear State-Space

Algorithm: We cast the mortality models in state-space form, and perform MCMC sampling of the model parameters. In the Bayesian setting, our aim is to conduct inference on the joint posterior density

$$\pi(\boldsymbol{\kappa}_{\mathbf{1:t_T}}, \boldsymbol{\Psi} | \boldsymbol{y}_{1:t_T}),$$

using Gibbs sampling (Geman and Geman, 1984),

$$\pi(\boldsymbol{\kappa}_{\mathbf{1:t_T}}, \boldsymbol{\Psi} | \boldsymbol{y}_{1:t_T}) \propto \pi(\boldsymbol{\Psi} | \boldsymbol{\kappa}_{1:t_T}, \boldsymbol{y}_{1:t_T})\ \pi(\boldsymbol{\kappa}_{1:t_T} | \boldsymbol{\Psi}, \boldsymbol{y}_{1:t_T}) \qquad (2)$$

1. Initialize $\boldsymbol{\Psi} = \boldsymbol{\Psi}^{(0)}$
2. For $i = 1, \ldots, M$,
   1. sample $\boldsymbol{\kappa}^{(i)}$ from $\pi(\boldsymbol{\kappa}_{1:t_T} | \boldsymbol{\Psi}^{(i-1)}, \boldsymbol{y}_{1:t_T})$ - Extended Kalman Filter
   2. sample $\boldsymbol{\Psi}^{(i)}$ from $\pi(\boldsymbol{\Psi} | \boldsymbol{\kappa}^{(i)}_{1:t_T}, \boldsymbol{y}_{1:t_T})$ - MCMC

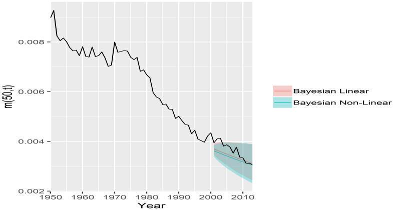


Figure 1: Age 50 UK

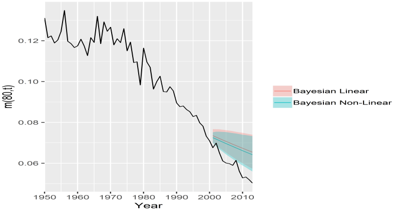


Figure 3: Age 80 AUS

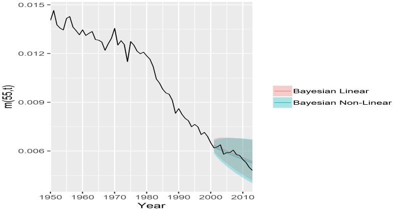


Figure 2: Age 55 UK

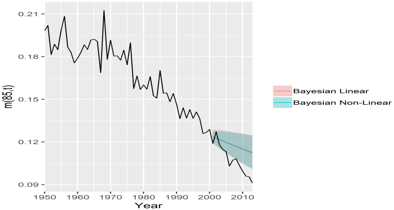


Figure 4: Age 85 AUS

Liability calculation

Let the survival rate of a person aged x surviving for the next t years be found by,

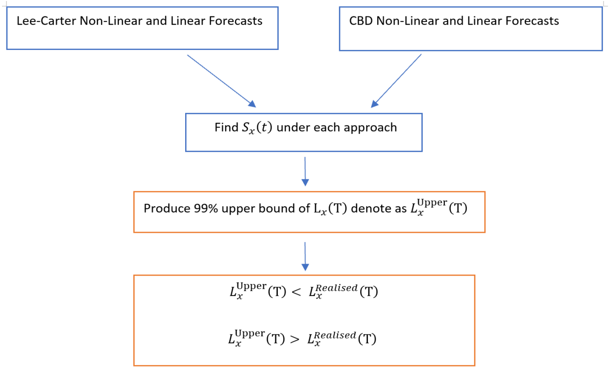
$$S_x(t) = \prod_{i=1}^{t}(1 - q_{x+i,t_1+i}) \qquad (3)$$

Assume now we are obligated to pay \$1 to a person currently aged x for the next T years. Let the price of a zero coupon bond which matures in t years be denoted as $P(0,t)$, we then have the liability for a \$1 annuity to a person aged x over T years to be,

$$L_x(T) = \sum_{t=1}^{T} P(0,t)S_x(t),$$

$$L_x^{\text{Realised}}(T) = \sum_{t=1}^{T} P(0,t)S_x^{\text{Realised}}(t),$$

$$L_x^{\text{Upper}}(T) = \sum_{t=1}^{T} P(0,t)S_x(t), \text{Upper 99\% Quantile for } S_x(t)$$



$$\text{"Hit" } \mathcal{I}_x(T) = \begin{cases} 0, & \text{for } \mathrm{L}_x^{\mathrm{Realised}}(T) < \mathrm{L}_x^{\mathrm{upper}}(T) \\ 1, & \text{for } \mathrm{L}_x^{\mathrm{Realised}}(T) > \mathrm{L}_x^{\mathrm{upper}}(T) \end{cases}$$

Here we apply a 5-year forecast horizon and obtain lower and upper bounds on $L_x(T)$ based on the Linear and Non-Linear estimation methods. The $L_x^{\text{Upper}}(T)$, represents $L_x(T)$ calculated at the 99% quantile of our mortality forecasts

Liability Bounds

Lee-Carter model

Country	Age	Bayesian Linear	Bayesian Non-Linear	Realised	$\mathcal{I}_{\text{age}}(5)$
AUS	60	(7.9589, 8.1067)	(7.9558, 8.1125)	8.0761	(0,0)
	65	(7.4510, 7.9092)	(7.3982, 7.8276)	7.7898	(0,0)
	75	(6.2556, 6.5839)	(6.2647, 6.6106)	6.5976	(1,0)
USA	60	(7.7743, 7.8864)	(7.4093, 7.5595)	7.8804	(0,0)
	65	(7.4023, 7.5428)	(7.4093, 7.5595)	7.5768	(1,1)
	75	(6.1483, 6.3428)	(6.1582, 6.3647)	6.4138	(1,1)

CBD model

Country	Age	Bayesian Linear	Bayesian Non-Linear	Realised	$\mathcal{I}_{\text{age}}(5)$
AUS	60	(7.9639, 8.1086)	(7.9695, 8.1138)	8.0761	(0,0)
	65	(7.5739, 7.8292)	(7.5674, 7.8261)	7.7898	(0,0)
	75	(6.0574, 6.7357)	(5.9730, 6.6861)	6.5976	(0,0)
USA	60	(7.7947, 7.9147)	(7.8104,7.9209)	7.8804	(0,0)
	65	(7.3881, 7.5661)	(7.3937, 7.5638)	7.5768	(1,1)
	75	(5.9729, 6.3445)	(5.9343, 6.2868)	6.4169	(1,1)

Backtesting VaR using Bayesian Decision Theory

Model 1: Null Hypothesis, 'hits' occur with probability $\alpha = \alpha^* = 0.01$.

$$L(\mathcal{I}|\alpha) = \alpha^{m_1}(1-\alpha)^{m-m_1}, \tag{4}$$

where m represents the sample size and m_1 represents the number of hits.

Model 2: Alternative Hypothesis, 'hits' occur with probability different from 0.01.

Given non-informative priors

$$\pi(\alpha) = \begin{cases} 1 & \text{if } \alpha = \alpha^*, \\ \text{Beta}(0.5, 0.5), & \text{if } \alpha \neq \alpha^*. \end{cases}$$

$$\text{BF}_{01} = \frac{(\alpha^*)^{m_1}(1-\alpha^*)^{m-m_1}}{\text{B}(m_1+0.5, m+0.5)} > 1$$

Table of Results

Table 1: Bayes Factor under Bayesian Linear and Bayesian Non-Linear methods

	Bayes Factor	
	Bayesian Linear	Bayesian Non-Linear
Lee-Carter	8.85×10^{-18}	2.91
CBD	1.02×10^{-7}	1.96

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- Cast two mortality models (LC and CBD) in non-linear state-space form.
- Compared the result to show that the non-linear state-space form succeeds the linear case under our backtesting framework.

The End

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