Supplementary Information: Bayesian Value-at-Risk backtesting: The case of annuity pricing

Melvern Leung[a], Youwei Li[b], Athanasios A. Pantelous[a,*], Samuel A. Vigne[c]

[a] *Department of Econometrics and Business Statistics, Monash Business School, Monash University, Australia*
[b] *Hull University Business School, University of Hull, U.K.*
[c] *Trinity Business School, The University of Dublin, Ireland*

Abstract

We propose a new Unconditional Coverage backtest for VaR-forecasts under a Bayesian framework that significantly minimise the direct and indirect effects of p-hacking or other biased outcomes in decision-making, in general. Especially, after the global financial crisis of 2007-09, regulatory demands from Basel III and Solvency II have required a more strict assessment setting for the internal financial risk models. Here, we employ linear and nonlinear Bayesianised variants of two renowned mortality models to put the proposed backtesting technique into the context of annuity pricing. In this regard, we explore whether the stressed longevity scenarios are enough to capture the experienced liability over the forecasted time horizon. Most importantly, we conclude that our Bayesian decision theoretic framework quantitatively produce a strength of evidence favouring one decision over the other.

Keywords: Decision analysis; Value-at-Risk; Backtesting; Bayesian framework; Longevity risk

[*] Corresponding author

Email addresses: Melvern.Leung@monash.edu (Melvern Leung), Youwei.Li@hull.ac.uk (Youwei Li), Athanasios.Pantelous@monash.edu (Athanasios A. Pantelous), vignes@tcd.ie (Samuel A. Vigne)

1. Mortality Modelling

State-space models allows a unified framework for treating a variety of problems in time series analysis (Durbin and Koopman, 2012). The idea behind state-space modelling is to obtain information about the unobserved state vector using information obtained from the measurement data. However, in many instances there may exist high serial correlation in time series data and thus routine computations become inefficient. This brings us to the idea of filtering and smoothing techniques which aims to improve computational efficiency in state-space models. With the introduction of the Kalman Filter (Kalman, 1960), state-space models have recently been used extensively in a variety of fields; especially in econometrics after the implementation of the Kalman Filter in Auto-Regressive Moving-Average (ARMA) models (Akaike, 1998). In this chapter, we aim to improve upon current state-of-the-art actuarial mortality models by casting these models in state-space form and implementing filtering and smoothing algorithms.

In recent years mortality improvements have started a worrying trend for many actuaries. These mortality improvements has exacerbated the liability involved when insurance companies sell annuity products that are contingent on the survival of the policy holder. Similarly, this issue exists for defined benefit plans which have ongoing payments contingent on the longevity of their members. To further emphasize the extent of this problem, the US life insurance business currently holds approximately $2 trillion in annuity reserves as of 2006, however their liability is estimated to be $6 trillion (ACLI[1], 2006). Historically, mortality rates were modeled in a deterministic fashion, and to some extent includes subjective considerations (Cairns et al., 2006c). However, recently there has been unpredictable and improving trends in longevity experiences, this situation entails current research to focus on developing stochastic mortality models to help improve the predictions for future mortality rates and more recently with the advancements in the econometric literature this has lead to the development of methods to improve model parameter estimation simultaneously (see Booth, 2006).

One well known stochastic mortality model is the Lee-Carter model proposed by Lee and Carter (1992), it models the log of death rates under both age and time factors. Due to its simplistic model structure, many have used the Lee-Carter model as a benchmark tool for population forecasts and uncover future mortality trends. The Lee-Carter model is also widely accepted and used by many in the context of mortality forecasting (see e.g. Li and Lee, 2005, Li et al., 2015, Booth et al., 2002, Denuit

[1] American Council-Life Insurers

et al., 2007). Furthermore because of it's popularity, there exists many adaptations and extensions to the Lee-Carter model, for instance, Renshaw et al. (1996) included a cohort term in the Lee-Carter model which tries to capture the cohort effects; a functional data approach extension by Hyndman and Ullah (2007), and additionally a p-spline approach by Currie et al. (2004). In 2005, Czado et al. (2005) introduced and pioneered a Bayesian approach to mortality modelling, where a Poisson error structure for death counts was used, this mainly stems from the recent establishments of the Markov Chain Monte Carlo(MCMC) simulation methods (see Chib and Jeliazkov (2001)) which allowed for easy sampling from posterior distributions which previously were thought to be computationally not viable. To extend on this, Pedroza (2006) kept the Lee-Carter model using Gaussian assumptions on the error term, and directly applied a Gibbs sampler alongside a Kalman Filter Backward smoother to retrieve their conditional posterior distribution samples. A few others who have applied this technique are Fung et al. (2017), Kogure et al. (2009) and Kogure and Kurachi (2010). Due to the large surmounting research based on the Lee-Carter model, we choose the Lee-Carter model as one of our benchmark models used throughout this thesis.

More recently in 2008, Cairns et al. (2006c) introduced the Cairns Blake Dowd(CBD) model, with the aim of not only to out-perform the standard Lee-Carter(LC) model in the mortality forecasting sense, but also be implemented in a mortality pricing framework. The CBD model was the next revolutionary step in mortality modelling, it is used frequently alongside the Lee-Carter model as a benchmark tool, not only for mortality forecasts but also used in financial applications. In the context of mortality pricing, Cairns et al. (2006c) originally applied the CBD model to derive a risk premium for the EIB longevity bond, where both Li (2010) and Li and Ng (2011) uses the CBD model as a comparison to other CBD model variants in deriving risk premiums for longevity under their proposed maximum entropy approach. Dowd et al. (2010b), Dowd et al. (2010b), Cairns et al. (2008), Cairns et al. (2011) and Cairns et al. (2009) provided forecast results that benchmark the CBD model and its other variants based on specific risk measures. Further generalizations to the CBD model can be found in Cairns et al. (2009), and Li (2010). The vast utilization of the CBD model steers us to choose it as our secondary mortality model benchmark.

In this section, we will use a Bayesian state-space framework applied to both the LC and CBD models with their respective linear and nonlinear counterparts, we take advantage of the Extended Kalman Filter (EKF) algorithm which is a one-step estimation of the latent states and the Gibbs sampler (cases with known distribution) or Random Walk Metropolis-Hastings (MH) (cases with an analytical form but no known distribution) for static parameters embedded in an MCMC.

1.1. Bayesian state-space model estimation

In a Bayesian model estimation framework we aim to conduct distributional inference on model parameters through relevant conditional distributions. In contrast, a frequentist approach aims to determine statistical inference based on properties of estimators, confidence intervals and hypothesis testing in a repeated sampling setting (Petris et al., 2009). The advantage of treating parameters as a probability distribution is that we are able to infer distributional properties of our underlying parameters which allows us to account for parameter uncertainty.

Assume we have a Hidden Markov Model (HMM) with static model parameters denoted by Θ and prior distribution $p(\Theta)$ and any parameters for the prior distributions are referred to as hyper-parameters (or a priori distribution parameter) in the Bayesian context. A latent process (hidden dynamics) denoted by $\mathbf{k_{1:T}}$, and the sample data denoted by $\mathbf{y_{1:T}}$. The likelihood function is denoted by $p(\mathbf{y_{1:T}}|\mathbf{\Theta})$. Then, the model dynamics can be represented as,

$$p(\mathbf{y_t}|\mathbf{k_{1:t}}, \mathbf{y_{1:t-1}}, \mathbf{\Theta}) = p(\mathbf{y_t}|\mathbf{k_t}, \mathbf{\Theta}), \tag{1.1}$$

$$p(\mathbf{k_{t+1}}|\mathbf{k_{1:t}}, \mathbf{y_{1:T}}, \mathbf{\Theta}) = p(\mathbf{k_{t+1}}|\mathbf{k_t}, \mathbf{y_t}, \mathbf{\Theta}). \tag{1.2}$$

Here, Eq. (1.1) is commonly known as the measurement equation, and Eq. (1.2) is known as the state transition equation. In most HMM models, the latent dynamics of Eq. (1.2) can be reduced to $p(\mathbf{k_{1:T}}|\mathbf{\Theta})$. Under a Bayesian approach, we wish to infer properties from the full posterior distribution of our unknown parameters, denoted by $p(\mathbf{k_{1:T}}, \mathbf{\Theta}|\mathbf{y_{1:T}})$. The full posterior distribution may be hard to compute numerically or analytically, but due to the imposed Markovian dynamics we can use the Gibbs sampler developed by Geman and Geman (1984). This technique allows the simulation of draws from the full posterior distribution by drawing from its constituent conditional distributions (see Casella and George (1992) and Geman and Geman (1984) for a detailed explanation of Gibbs sampling). The full posterior distribution can be broken down in the following way,

$$p(\mathbf{k_{1:T}}, \mathbf{\Theta}|\mathbf{y_{1:T}}) \propto \underbrace{p(\mathbf{k_{1:T}}|\mathbf{y_{1:T}}, \mathbf{\Theta})}_{(A)} \underbrace{p(\Theta|\mathbf{k_{1:T}}, \mathbf{y_{1:T}})}_{(B)},$$

where,

$$p(\mathbf{k_{1:T}}|\mathbf{\Theta}, \mathbf{y_{1:T}}) \propto p(\mathbf{y_{1:T}}|\mathbf{k_{1:T}}, \mathbf{\Theta}) \ \mathbf{p}(\mathbf{k_{1:T}}|\mathbf{\Theta}), \tag{A}$$

$$p(\Theta|\mathbf{k_{1:T}}, \mathbf{y_{1:T}}) \propto p(\mathbf{y_{1:T}}|\mathbf{k_{1:T}}, \mathbf{\Theta}) \ \mathbf{p}(\mathbf{k_{1:T}}|\mathbf{\Theta}) \ \mathbf{p}(\mathbf{\Theta}). \tag{B}$$

The MCMC algorithm allows the easy computation of high dimensional data by breaking down the joint model posterior distribution into its constituent conditional distribution (for more information see Geweke et al. (2011)). A filtering algorithm is usually used for Eq. (A), in the case of linear Gaussian measurement and state equations, the Kalman Filter can be used, whereas for the nonlinear case the EKF can be applied instead. These Filtering algorithm allows for a one-step estimation procedure for the latent dynamics, and will requires less computation time. Embedded in this MCMC algorithm is the Gibbs sampler for the static parameters in Eq. (B), for cases where parameters did not have a conjugate form, a MH algorithm was added.

1.2. The Lee-Carter model

One well known stochastic mortality model is the LC model proposed by Lee and Carter (1992), it models the log of death rates under two-factors (both age and time). Due to its simplistic model structure, many have used the LC model as a bench-marking tool to test for population forecasts or future mortality trends; the LC model is also widely accepted and used by many in the context of mortality forecasting, for example by Booth et al. (2002), Li and Lee (2005), Li et al. (2015) and Denuit et al. (2007). Furthermore because of it's popularity, there have been many adaptations and extensions to the LC model from its original, for instance, Renshaw et al. (1996) included a cohort term in the LC model which tries to capture the cohort effects, that is the effects that a population currently aged x will have on the population aged $x+1$ in one years time; a functional data approach extension by Hyndman and Ullah (2007), and also a p-spline approach by Currie et al. (2004). In 2005, Czado et al. (2005) introduced and pioneered a Bayesian approach to mortality modelling, where a Poisson Error structure for death rates is used, this mainly stems from the recent establishments of the MCMC simulation methods (for more information see Chib and Jeliazkov (2001)), which allowed for easy sampling from posterior distributions which previously were thought to be computationally not viable, however one caveat to this approach was that the dynamics of the latent states was ignored and each latent variable was independently sampled using a MH algorithm. To extend on this, Pedroza (2006) continued the LC model using Gaussian assumptions on the error term allowing the applicability of a linear-state space model, the estimation was done through a Gibbs sampler alongside a Kalman Filter Backward smoother to retrieve their conditional posterior distribution for the parameters, however one shortfall for this approach is that it conditioned on the crude mortality rates for the observations. A few others such as Kogure et al. (2009) and Kogure and Kurachi (2010) have also applied this technique in their paper.

The standard Poisson Regression on death rates uses a two step estimation procedure and bypasses the conditioning on the crude mortality estimates. Firstly, the Maximum Likelihood Estimator (MLE) on a Poisson likelihood, is used on the unobservable latent states, then secondly, the estimated latent states is fitted to an ARIMA model which potential forecasts are based upon. On the contrary, the idea behind the state-space framework is that it aims to develop knowledge about a latent states, using information from the observed time series, while maintaining the underlying structure of the latent dynamics. This is a one-step estimation procedure, in contrast to the two-step Poisson MLE. As part of our analysis, we incorporate a Bayesian state-space framework. The advantages of using a Bayesian approach is that we can derive the posterior distribution and infer sampling properties for our static and latent parameters.

The Poisson Regression model has been applied in the mortality modelling literature (see, Brouhns et al., 2002, 2005, Delwarde et al., 2007, Millossovich et al., 2017). More importantly, the Longevity Life and Markets Associations (LLMA) uses this estimation procedure when quantifying longevity risk with its applications to standardize a longevity market[2]. In essence, they are a non-for-profit organization aiming to standardise a longevity index which market participants can base their products on. The usefulness of the Poisson Maximum likelihood estimation is that it models the Death counts rather than the crude mortality rate. More specifically under a Bayesian framework Czado et al. (2005) uses a Gibbs sampler and a Random Walk Metropolis-Hastings embedded in an MCMC algorithm. Due to the non-linearity of the Poisson Regression, frequentist approach relies heavily on maximizing the likelihood function, in contrast Bayesian estimation has focused on a Random-Walk Metropolis-Hastings (RWMH) algorithm. To our knowledge, the Poisson Regression has not been cast in state-space form previously, and in our case we utilize the EKF for an approximation to the latent dynamics, then conditional on the approximation we run a MH algorithm. The difference between our method, and Czado et al. (2005) is that we will not rely on a RWMH algorithm which requires an extra tuning parameter, instead our candidate will be taken from the EKF step.

1.3. Lee-Carter model: Nonlinear variant posterior derivations

Lemma 1.1. *The posterior distribution for α_x has a LogGamma kernal, and is given by,*

$$\pi(\alpha_x|\boldsymbol{y}, \boldsymbol{\beta}, \boldsymbol{\kappa}) \sim LogGamma\left(b_\alpha + \sum_{t=1}^{T} E_{x,t}\exp(\beta_x\kappa_t), d_{x,t} + a_\alpha\right).$$

Proof. To obtain the conditional posterior distribution for α_x, we begin with the likelihood function and prior distribution. The next step is to determine if the conditional posterior distribution kernel

[2]LLMA is a company funded by investment banks and insurance companies with the aim of standardizing the longevity markets.

comes from a known probability distribution.

$$
\begin{aligned}
\prod_{t=1}^{T} l(y_{x,t}|\kappa_t, \alpha_x, \boldsymbol{\beta}, \delta, \sigma_\omega^2, \sigma_\beta^2) &\propto \prod_{t=1}^{T} \exp\left(-\mathrm{E}_{x,t} m_{x,t}\right) \left(\mathrm{E}_{x,t} m_{x,t}\right)^{d_{x,t}}, \\
&\propto \prod_{t=1}^{T} \exp\left(-\mathrm{E}_{x,t} \exp(\alpha_x + \beta_x \kappa_t)\right) \exp\left(d_{x,t} \log(\mathrm{E}_{x,t} m_{x,t})\right), \\
&\propto \prod_{t=1}^{T} \exp\left(-\mathrm{E}_{x,t} \exp(\alpha_x + \beta_x \kappa_t)\right) \exp\left(d_{x,t}(\alpha_x + \beta_x \kappa_t)\right), \\
&\propto \prod_{t=1}^{T} \exp\left(-\mathrm{E}_{x,t} \exp(\beta_x \kappa_t) \exp(\alpha_x)\right) \exp(\alpha_x)^{d_{x,t}}.
\end{aligned}
$$

Combining the likelihood function and the prior distribution for α_x, $\pi(\alpha_x) \sim \text{LogGamma}(a_\alpha, b_\alpha)$, the conditional posterior can be found by,

$$
\begin{aligned}
\pi(\alpha_x|\boldsymbol{y}, \boldsymbol{\beta}, \boldsymbol{\kappa}, \delta, \sigma_\omega^2, \sigma_\beta^2) &\propto \prod_{t=1}^{T} l(y_{x,t}|\kappa_t, \alpha_x, \boldsymbol{\beta}, \delta, \sigma_\omega^2, \sigma_\beta^2)\pi(\alpha_x), \\
&\propto \exp\left(-\exp(\alpha_x) \sum_{t=1}^{T} \mathrm{E}_{x,t} \exp(\beta_x \kappa_t)\right) \exp(\alpha_x)^{d_{x,t}} (\exp(\alpha_x)^{a_\alpha - 1}) \exp(-b_\alpha \exp(\alpha_x)), \\
&\propto \exp\left(-\exp(\alpha_x)(b_\alpha + \sum_{t=1}^{T} \mathrm{E}_{x,t} \exp(\beta_x \kappa_t))\right) (\exp(\alpha_x)^{d_{x,t} + a_\alpha - 1}).
\end{aligned}
$$

Thus the posterior for α_x has a LogGamma kernel with parameters,

$$
\pi(\alpha_x|\boldsymbol{y}, \boldsymbol{\kappa}, \boldsymbol{\Psi}) \sim \text{LogGamma}\left(b_\alpha + \sum_{t=1}^{T} \mathrm{E}_{x,t} \exp(\beta_x \kappa_t), d_{x,t} + a_\alpha\right).
$$

□

Lemma 1.2. *The closed form expression for the posterior of β_x is given by,*

$$
\pi(\beta_x|\boldsymbol{y}, \kappa_t, \boldsymbol{\alpha}, \boldsymbol{\beta}, \delta, \sigma_\omega^2, \sigma_\beta^2) \propto \exp\left(-\sum_{t=1}^{T} E_{x,t} \exp(\beta_x \kappa_t) \exp(\alpha_x) - \frac{\beta_x^2}{2\sigma_\beta^2}\right) \exp\left(\beta_x \kappa_t\right)^{d_{x,t}}.
$$

Proof. To draw β_x we use a MH algorithm, since there is no known family of distributions for the posterior distribution of β_x. To apply the MH algorithm we need the closed form posterior distribution of β_x.

$$
\prod_{t=1}^{T} l(y_{x,t}|\kappa_t, \boldsymbol{\alpha}, \boldsymbol{\beta}, \delta, \sigma_\omega^2, \sigma_\beta^2) \propto \prod_{t=1}^{T} \exp\left(-\mathrm{E}_{x,t} \exp(\alpha_x + \beta_x \kappa_t)\right) \exp\left(d_{x,t}(\beta_x \kappa_t)\right),
$$

We combine the likelihood function for β_x and the prior distribution $\pi(\beta_x) \sim N(\mu_\beta, \sigma_\beta^2)$ to obtain the posterior distribution in closed form.

$$\begin{aligned}
\pi(\beta_x|\boldsymbol{y}, \kappa_t, \boldsymbol{\alpha}, \boldsymbol{\beta}, \delta, \sigma_\omega^2, \sigma_\beta^2) \propto & \prod_{t=1}^{T} l(y_{x,t}|\kappa_t, \alpha_x, \boldsymbol{\beta}, \delta, \sigma_\omega^2, \sigma_\beta^2)\pi(\beta_x), \\
& \propto \exp\left(-\sum_{t=1}^{T} \mathrm{E}_{x,t}\exp(\beta_x\kappa_t)\exp(\alpha_x)\right)\exp(\beta_x\kappa_t)^{d_{x,t}}\exp\left(\frac{-\beta_x^2}{2\sigma_\beta^2}\right), \\
& \propto \exp\left(-\sum_{t=1}^{T} \mathrm{E}_{x,t}\exp(\beta_x\kappa_t)\exp(\alpha_x) - \frac{\beta_x^2}{2\sigma_\beta^2}\right)\exp\left(\beta_x\kappa_t\right)^{d_{x,t}}.
\end{aligned}$$

□

Lemma 1.3. *The posterior distribution of σ_β^2 is given by,*

$$\pi(\sigma_\beta^2|\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}, \delta, \sigma_\omega^2) \propto I.G\left(a_\beta + \tfrac{N}{2}, b_\beta + \tfrac{1}{2}\boldsymbol{\beta}\boldsymbol{\beta}'\right).$$

Proof. Combining the likelihood function and prior distribution $\pi(\sigma_\beta^2) \sim I.G(a_\sigma, b_\sigma)$,

$$\begin{aligned}
\pi(\sigma_\beta^2|\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}) \propto & \prod_{x=x_1}^{x_n}\prod_{t=1}^{T} l(\beta_x|\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\kappa}, \delta, \sigma_\omega^2, \sigma_\beta^2)\pi(\sigma_\beta^2), \\
& \propto \prod_{x=x_1}^{x_n} \exp\left(-\sum_{t=1}^{T} \mathrm{E}_{x,t}\exp(\beta_x\kappa_t)\exp(\alpha_x) - \frac{\beta_x^2}{2\sigma_\beta^2}\right), \\
& \times \exp(\beta_x\kappa_t)^{d_{x,t}}(\sigma_\beta^2)^{-a_\sigma - 1}\exp\left(\frac{-b_\sigma}{\sigma_\beta^2}\right), \\
& \propto (\sigma_\beta^2)^{-a_\sigma + \frac{n}{2} - 1}\exp\left(\frac{-b_\sigma + \frac{1}{2}\boldsymbol{\beta}\boldsymbol{\beta}'}{\sigma_\beta^2}\right).
\end{aligned}$$

Thus $\pi(\sigma_\beta^2|\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}, \delta, \sigma_\omega^2)$ has an Inverse Gamma distribution with parameters,

$$\pi(\sigma_\beta^2|\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}, \delta, \sigma_\omega^2) \propto \mathrm{I.G}\left(a_\beta + \tfrac{N}{2}, b_\beta + \tfrac{1}{2}\boldsymbol{\beta}\boldsymbol{\beta}'\right).$$

□

Lemma 1.4. *The posterior distribution for δ is given by,*

$$\pi(\delta|\boldsymbol{y}, \boldsymbol{\kappa}, \sigma_\omega^2) \sim N\left(\frac{\mu_\delta\sigma_\omega^2 + \sigma_\delta^2\sum_{t=1}^{T}(\kappa_t - \kappa_{t-1})}{\sigma_\delta^2\sigma_\omega^2}, \ \frac{\sigma_\delta^2\sigma_\omega^2}{T\sigma_\delta^2 + \sigma_\omega^2}\right).$$

Proof. Combining the likelihood function and prior distribution $\pi(\delta) \sim N(\mu_\delta, \Sigma_\delta)$,

$$
\begin{aligned}
\pi(\delta|\boldsymbol{y}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}) &\propto \prod_{x=x_1}^{x_n} \prod_{t=1}^{T} l(y_{x,t}|\kappa_t, \alpha_x, \boldsymbol{\beta}, \delta, \sigma_\omega^2, \sigma_\beta^2)\pi(\boldsymbol{\kappa}|\boldsymbol{y}, \delta, \sigma_\omega^2)\pi(\delta|\boldsymbol{\kappa}, \sigma_\omega^2), \\
&\propto \prod_{t=1}^{T} \pi(\boldsymbol{\kappa}|\boldsymbol{y}, \delta, \sigma_\omega^2)\pi(\delta), \\
&\propto \prod_{t=1}^{T} \pi(\kappa_t|\kappa_{t-1}, \delta, \sigma_\omega^2)\pi(\delta), \\
&= \prod_{t=1}^{T} \frac{1}{\sigma_\delta\sqrt{2\pi}} e^{-(\delta-(\mu_\delta))^2/2\sigma_\delta^2} \times \frac{1}{\sigma_\omega\sqrt{2\pi}} e^{-(\kappa_t-(\kappa_{t-1}+\delta))^2/2\sigma_\omega^2}, \\
&\propto \exp\left(\frac{-1}{2}\left(\frac{(T\sigma_\delta^2+\sigma_\omega^2)\delta^2 - 2\delta(\mu_\delta\sigma_\omega^2 + \sigma_\delta^2\sum_{t=1}^{T}(\kappa_t-\kappa_{t-1}))}{\sigma_\delta^2\sigma_\omega^2}\right)\right).
\end{aligned}
$$

This has a normal distribution kernel, with parameters given by,

$$
\pi(\delta|\boldsymbol{y}, \boldsymbol{\kappa}, \sigma_\omega^2) \sim N\left(\frac{\mu_\delta\sigma_\omega^2 + \sigma_\delta^2\sum_{t=1}^{T}(\kappa_t-\kappa_{t-1})}{\sigma_\delta^2\sigma_\omega^2},\ \frac{\sigma_\delta^2\sigma_\omega^2}{T\sigma_\delta^2+\sigma_\omega^2}\right).
$$

□

Lemma 1.5. *The posterior distribution for σ_ω^2 is given by,*

$$
\pi(\sigma_\omega^2|\boldsymbol{y}, \boldsymbol{\kappa}, \delta) \sim I.G\left(a_\omega + \frac{T}{2},\ b_\omega + \frac{1}{2}\sum_{t=1}^{T}(\kappa_t - (\kappa_{t-1}+\delta))^2\right).
$$

Proof. Combining the likelihood function and prior distribution $\pi(\sigma_\omega^2) \sim IG(a_\omega, \beta_\omega)$,

$$
\begin{aligned}
\pi(\sigma_\omega^2|\boldsymbol{y}, \boldsymbol{\kappa}, \delta) &\propto \prod_{x=x_1}^{x_n} \prod_{t=1}^{T} l(y_{x,t}|\kappa_t, \alpha_x, \boldsymbol{\beta}, \delta, \sigma_\omega^2, \sigma_\beta^2)\pi(\boldsymbol{\kappa}|\boldsymbol{y}, \delta, \sigma_\omega^2)\pi(\sigma_\omega^2), \\
&\propto \prod_{t=1}^{T} \pi(\boldsymbol{\kappa}|\boldsymbol{y}, \delta, \sigma_\omega^2)\pi(\sigma_\omega^2), \\
&\propto \prod_{t=1}^{T} \pi(\kappa_t|\kappa_{t-1}, \boldsymbol{y}, \delta, \sigma_\omega^2)\pi(\sigma_\omega^2), \\
&= \prod_{t=1}^{T} \frac{1}{\sigma_\omega\sqrt{2\pi}} e^{-(\kappa_t-(\kappa_{t-1}+\theta))^2/2\sigma_\omega^2} \times \frac{1}{b_\omega^{-a_\omega}}(\sigma_\omega^2)^{-a_\omega-1}\exp(-\frac{1}{b_\omega\sigma_\omega^2}), \\
&\propto \frac{1}{(\sigma_\omega^2)^{(\frac{T}{2}+a_\omega+1)}} \exp\left(\frac{-1}{\sigma_\omega^2}\left(b_\omega + \frac{1}{2}\sum_{t=1}^{T}(\kappa_t-(\kappa_{t-1}+\delta))^2\right)\right).
\end{aligned}
$$

This has an Inverse Gamma distribution kernel, with parameters given by,

$$\pi(\sigma^2_\omega|\boldsymbol{y},\boldsymbol{\kappa},\delta) \sim I.G\left(a_\omega + \frac{T}{2},\ b_\omega + \frac{1}{2}\sum_{t=1}^{T}(\kappa_t - (\kappa_{t-1}+\delta))^2\right). \tag{1.3}$$

□

Algorithm 1 Extended Kalman Filter for LC Poisson Model

```
function EKF_κt(α, β, δ, σ²_ω)
Filter step
    for t = 1, ..., T do
        Initiate:
a_{0|0} = [0; 0] and P_{0|0} = [100 0; 0 100].
F_t = [log(E_t) + α   β]
G_t = [0 0; 0 1]
        Predictive:
a_{t|t-1} = G_t a_{t-1|t-1} + [1; δ]
P_{t|t-1} = G_t P_{t-1|t-1} G'_t + [0 0; 0 σ²_ω].
        Filter:
λ_t = F'_t a_{t|t-1}, Ŷ_t = exp(λ_t) d_t + λ_t − ι_n, V̂_t = exp(−λ_t)
Q_t = F_t P_{t|t-1} F'_t + diag(V̂_t), A_t = P_{t|t-1} F'_t Q_t^{-1}.
a_{t|t} = a_{t|t-1} + A_t (Ŷ_t − λ_t)
P_{t|t} = (𝕀_2 − A_t F_t) P_{t|t-1}.
    end for
Backward Smoothing step
    for t = T − 1, ..., 1 do
        h_t = a_{t|t} + P_{t|t} G_t P^{-1}_{t|t-1} (h_{t+1} − G_t a_{t+1|t+1})
        H_t = P_{t|t} + P_{t|t} G_t P^{-1}_{t|t-1} (H_{t+1} − P_{t|t}) (P_{t|t} G_t P^{-1}_{t|t-1})'.
        Sample k_t ∼ N(h_t, H_t).
    end for
end function
```

Algorithm 2 MH for κ_t under nonlinear LC model

function MH-LC_$\kappa_t(\boldsymbol{\kappa}^*, \boldsymbol{\kappa}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \delta, \sigma_\omega^2)$
  for $t = 1, ..., T$ **do**
    if $1 < t < T$ **then**
      $f = \exp\left(\sum_x(-\mathrm{E}_{x,t}\exp(\alpha_x + \beta_x\kappa_t)) + \sum_x(\beta_x\kappa_t\mathrm{d}_{x,t}) - \frac{1}{2\sigma_\omega^2}(\kappa_t - \delta)^2 - \frac{1}{2\sigma_\omega^2}(\kappa_{t+1} - \delta)^2\right)$
      $f^* = \exp\left(\sum_x(-\mathrm{E}_{x,t}\exp(\alpha_x + \beta_x\kappa_t^*)) + \sum_x(\beta_x\kappa_t^*\mathrm{d}_{x,t}) - \frac{1}{2\sigma_\omega^2}(\kappa_t^* - \delta)^2 - \frac{1}{2\sigma_\omega^2}(\kappa_{t+1}^* - \delta)^2\right)$
      $\phi = \min(f^*/f, 1)$
    else
      $f = \exp\left(\sum_x(-\mathrm{E}_{x,t}\exp(\alpha_x + \beta_x\kappa_t)) + \sum_x(\beta_x\kappa_t\mathrm{d}_{x,t}) - \frac{1}{2\sigma_\omega^2}(\kappa_t - \delta)^2\right)$
      $f^* = \exp\left(\sum_x(-\mathrm{E}_{x,tt}\exp(\alpha_x + \beta_x\kappa_t^*)) + \sum_x(\beta_x\kappa_t^*\mathrm{d}_{x,t}) - \frac{1}{2\sigma_\omega^2}(\kappa_t^* - \delta)^2\right)$
      $\phi = \min(f^*/f, 1)$
    end if
    $u \sim U(0, 1)$
    if $u < \phi$ **then**
      $\hat{\kappa}_t = \kappa_t^*$
    else
      $\hat{\kappa}_t = \kappa_t$
    end if
  end for
  $\hat{\boldsymbol{\kappa}} = \hat{\boldsymbol{\kappa}} - (1/T)\sum_t \hat{\kappa}_t$
end function

Algorithm 3 MH for $\boldsymbol{\beta}$ under nonlinear LC model

function MH-LC_$\boldsymbol{\beta}(\boldsymbol{\kappa}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \sigma_\omega^2, \sigma_\beta^2)$
  for $x = x_1, ..., x_n$ **do**
    Draw $\beta_x^* \sim N(0, \sigma_x^2)$
    $f = \exp\left(\sum_t(\mathrm{E}_{x,t}\exp(\alpha_x + \beta_x\kappa_t)) - \sum_t(\beta\kappa_t\mathrm{d}_{x,t}) + \frac{\beta_x^2}{2\sigma_\beta}\right)$
    $f^* = \exp\left(\sum_t(\mathrm{E}_{x,t}\exp(\alpha_x + \beta_x^*\kappa_t)) - \sum_t(\beta_x^*\kappa_t\mathrm{d}_{x,t}) + \frac{(\beta_x^*)^2}{2\sigma_\beta}\right)$
    $\phi = \min(f^*/f, 1)$
    $u \sim U(0, 1)$
    if $u < \phi$ **then**
      $\hat{\beta}_x = \beta_x^*$
    else
      $\hat{\beta}_x = \beta_x$
    end if
  end for
  $\hat{\boldsymbol{\beta}} = \frac{\hat{\boldsymbol{\beta}}}{\sum_x \hat{\beta}_x}$
end function

1.4. The Cairns-Blake-Dowd model

In 2008, Cairns et al. (2006a) introduced the CBD model, with the aim of not only to out-perform the standard LC model in the mortality forecasting sense, but also be implemented in a mortality pricing framework (Cairns et al., 2006b). This model was the next revolutionary step in mortality modelling due to the strong performance in forecasting mortality rates for older population, it is used frequently alongside the LC model as a bench-marking tool, not only for mortality forecasts but from a financial aspect. In the context of mortality pricing, Cairns et al. (2006a) originally applied the CBD model to derive a risk premium for the EIB longevity bond, where both Li (2010) and Li and Ng (2011) uses the model as a comparison to its other variants in deriving risk premiums for longevity under their proposed maximum entropy approach. In terms of testing and comparing the CBD model against other stochastic models, Dowd et al. (2010a), Dowd et al. (2010b), Cairns et al. (2008), Cairns et al. (2011) and Cairns et al. (2009) provided forecast results for many stochastic models and also determined which model out performed for a given risk measure. On a different note, Chan et al. (2014) created a longevity index based on the CBD model. Other generalizations to the model can be found in Cairns et al. (2009), and Li (2010).

The vast utilization of the CBD model steers us to choosing it as our secondary mortality model benchmark. We consider two approaches to the CBD model, firstly, we have the linear variant, where the model is cast in a linear state-space form with non-trivial Gaussian error structure. For the second model, we consider a Binomial distribution for the death counts and recover the latent dynamics of the CBD model using a combination of the EKF and MH.

1.5. CBD model Posterior Derivations

Lemma 1.6. *The closed form posterior distribution for $\boldsymbol{\kappa_t}$ is given by,*

For $t > 1$

$$\pi(\boldsymbol{\kappa_t}|\boldsymbol{\kappa_{t-1}}, \boldsymbol{y_t}, \boldsymbol{\theta}, \Sigma) \propto \exp\left(\sum_{x=x_1}^{x_n} \left(d_{x,t}(\kappa_{1,t} + (x-\bar{x})\kappa_{2,t}) - E_{x,t}\log\left(1 - e^{\kappa_{1,t}+(x-\bar{x})\kappa_{2,t}}\right)\right)\right.$$
$$\times \exp\left(-[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]\Sigma[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]'\right).$$

For $t = 1$

$$\pi(\boldsymbol{\kappa_t}|\boldsymbol{\kappa_{t-1}}, \boldsymbol{y_t}, \boldsymbol{\theta}, \Sigma) \propto \exp\left(\sum_{x=x_1}^{x_n} \left(d_{x,t}(\kappa_{1,t} + (x-\bar{x})\kappa_{2,t}) - E_{x,t}\log\left(1 - e^{\kappa_{1,t}+(x-\bar{x})\kappa_{2,t}}\right)\right)\right),$$
$$\times \exp\left(-[\boldsymbol{\kappa_t} - \boldsymbol{\theta}]\Sigma[\boldsymbol{\kappa_t} - \boldsymbol{\theta}]'\right).$$

Proof. To produce draws for $\boldsymbol{\kappa_t}$, we use the MH algorithm since there is no known distribution for the conditional posterior distribution for $\boldsymbol{\kappa_t}$ under a Gaussian prior. Since the conditioning is on $\boldsymbol{\kappa_{t-1}}$, there will be two separate posteriors that is when, $t = 1$ and $t > 1$. Thus,

for $t = 1$,

$$
\begin{aligned}
\pi(\boldsymbol{\kappa_t}|\boldsymbol{\kappa_{t-1}}, \boldsymbol{y_t}, \boldsymbol{\theta}, \Sigma) \propto & l(\boldsymbol{y_t}|\boldsymbol{\kappa_t})\pi(\boldsymbol{\kappa_t}|\boldsymbol{\kappa_{t-1}}, \boldsymbol{\theta}, \Sigma) \\
\propto & \prod_{x=x_1}^{x_n} \binom{\mathrm{E}_{x,t}}{\mathrm{d}_{x,t}} q_{x,t}^{\mathrm{d}_{x,t}} (1-q_{x,t})^{\mathrm{E}_{x,t}-\mathrm{d}_{x,t}} \exp\left(-[\boldsymbol{\kappa_t} - \boldsymbol{\theta}]\Sigma[\boldsymbol{\kappa_t} - \boldsymbol{\theta}]'\right), \\
\propto & \prod_{x=x_1}^{x_n} \left(\frac{q_{x,t}}{1-q_{x,t}}\right)^{\mathrm{d}_{x,t}} (1-q_{x,t})^{\mathrm{E}_{x,t}} \exp\left(-[\boldsymbol{\kappa_t} - \boldsymbol{\theta}]\Sigma[\boldsymbol{\kappa_t} - \boldsymbol{\theta}]'\right), \\
\propto & \exp\left(\sum_{x=x_1}^{x_n} (\mathrm{d}_{x,t}(\kappa_{1,t} + (x-\bar{x})\kappa_{2,t}) - \mathrm{E}_{x,t} \log\left(1 - e^{\kappa_{1,t}+(x-\bar{x})\kappa_{2,t}}\right)\right), \\
\times & \exp\left(-[\boldsymbol{\kappa_t} - \boldsymbol{\theta}]\Sigma[\boldsymbol{\kappa_t} - \boldsymbol{\theta}]'\right).
\end{aligned}
$$

for $t > 1$,

$$
\begin{aligned}
\pi(\boldsymbol{\kappa_t}|\boldsymbol{\kappa_{t-1}}, \boldsymbol{y_t}, \boldsymbol{\theta}, \Sigma) \propto & l(\boldsymbol{y_t}|\boldsymbol{\kappa_t})\pi(\boldsymbol{\kappa_t}|\boldsymbol{\kappa_{t-1}}, \boldsymbol{y_t}, \boldsymbol{\theta}, \Sigma) \\
\propto & \prod_{x=x_1}^{x_n} \binom{\mathrm{E}_{x,t}}{\mathrm{d}_{x,t}} q_{x,t}^{\mathrm{d}_{x,t}} (1-q_{x,t})^{\mathrm{E}_{x,t}-\mathrm{d}_{x,t}} \exp\left(-[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]\Sigma[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]'\right), \\
\propto & \prod_{x=x_1}^{x_n} \left(\frac{q_{x,t}}{1-q_{x,t}}\right)^{\mathrm{d}_{x,t}} (1-q_{x,t})^{\mathrm{E}_{x,t}} \exp\left(-[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]\Sigma[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]'\right), \\
\propto & \exp\left(\sum_{x=x_1}^{x_n} \sum_{t=1}^{T} (\mathrm{d}_{x,t}(\kappa_{1,t} + (x-\bar{x})\kappa_{2,t}) - \mathrm{E}_{x,t} \log\left(1 - e^{\kappa_{1,t}+(x-\bar{x})\kappa_{2,t}}\right)\right), \\
\times & \exp\left(-[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]\Sigma[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]'\right).
\end{aligned}
$$

□

Lemma 1.7. *The posterior distribution for $\boldsymbol{\theta}$ is given by,*

$$\pi(\boldsymbol{\theta}|\boldsymbol{y}, \boldsymbol{\kappa}, \Sigma) \sim N(\mu_\theta^p, \Sigma_\theta^p)$$

where,

$$
\begin{aligned}
\mu_\theta^p =& (\Sigma_\theta^{-1} + T\Sigma^{-1})^{-1} \left(\Sigma_\theta{}^{-1}\mu_\theta + \Sigma^{-1} \sum_{t=1}^{T} (\boldsymbol{\kappa_t} - \boldsymbol{\kappa_{t-1}})\right), \\
\Sigma_\theta^p =& \left(\Sigma_\theta^{-1} + T\Sigma^{-1}\right)^{-1}.
\end{aligned}
$$

Proof. Combining the likelihood function and prior distribution $\pi(\boldsymbol{\theta}) \sim N(\mu_\theta, \sigma_\theta^2)$,

$$
\begin{aligned}
\pi(\boldsymbol{\theta}|\boldsymbol{y}, \boldsymbol{\kappa}, \Sigma) \propto & \prod_{x=x_1}^{x_n} \prod_{t=1}^{T} l(\boldsymbol{y}|\boldsymbol{\kappa}, \boldsymbol{\theta}, \Sigma)\pi(\boldsymbol{\kappa_t}|\Sigma, \boldsymbol{\theta})\pi(\boldsymbol{\theta}), \\
\propto & \prod_{t=1}^{T} \pi(\boldsymbol{\kappa_t}|\boldsymbol{\kappa_{t-1}}, \Sigma)\pi(\boldsymbol{\theta}), \\
= & \prod_{t=1}^{T} \frac{1}{\sqrt{|\Sigma|(2\pi)^2}} e^{\left(\frac{-1}{2}[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]\Sigma^{-1}[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]'\right)} \times \\
& \frac{1}{\sqrt{|\Sigma_\theta|(2\pi)^2}} e^{\left(\frac{-1}{2}[\boldsymbol{\theta} - \boldsymbol{\mu_\theta}]\Sigma_\theta{}^{-1}[\boldsymbol{\theta} - \boldsymbol{\mu_\theta}]'\right)}, \\
\propto & \exp\left(\frac{-1}{2}\left(\boldsymbol{\theta}'(\Sigma_\theta + T\Sigma)\theta - 2\boldsymbol{\theta}^T(\boldsymbol{\mu_\theta}\Sigma + \Sigma^{-1}\sum_{t=1}^{T}(\boldsymbol{\kappa_t} - \boldsymbol{\kappa_{t-1}})\right)\right).
\end{aligned}
$$

The conditional posterior for $\boldsymbol{\theta}$ has a Normal distribution Kernel with the following parameters,

$$
\pi(\boldsymbol{\theta}|\boldsymbol{\kappa}, \Sigma) \sim N(\mu_\theta^p, \Sigma_\theta^p)
$$

where,

$$
\begin{aligned}
\mu_\theta^p = & (\Sigma_\theta^{-1} + T\Sigma^{-1})^{-1}\left(\Sigma_\theta{}^{-1}\mu_\theta + \Sigma^{-1}\sum_{t=1}^{T}(\boldsymbol{\kappa_t} - \boldsymbol{\kappa_{t-1}})\right), \\
\Sigma_\theta^p = & \left(\Sigma_\theta^{-1} + T\Sigma^{-1}\right)^{-1}.
\end{aligned}
$$

□

The posterior distribution for Σ has been known to have issues when the variance are small, and the correlation is close to 0. Thus we use the same Bayesian hierarchical structure as Leung et al. (2018) for the variance-covariance matrix of the latent dynamics. We first assign priors on the diagonal component of Σ and denote this as $\pi(\sigma_k^2)$ for $k \in (1, 2)$. Then, draw σ_k^2 from the conditional posterior $\pi(\sigma_k^2|\Sigma)$ and draw Σ from $\pi(\Sigma|\boldsymbol{y}, \boldsymbol{\kappa}, \boldsymbol{\theta}, \Sigma)$.

Lemma 1.8. *The posterior distribution for $\sigma_k^2 \quad k \in (1, 2)$ is given by,*

$$
\pi(\sigma_k^2|\Sigma) \sim I.G\left(\frac{\nu + 2}{2}, \nu(\Sigma_\omega^{-1})_{kk} + \frac{1}{A_k}\right) \text{ for } k \in (1, 2),
$$

The posterior distribution for Σ conditional on σ_k^2 is given by,

$$
\pi(\Sigma|\boldsymbol{y}, \boldsymbol{\kappa}, \boldsymbol{\theta}, \Sigma) \sim I.W\left(T + \nu,\ \psi + \sum_{t=1}^{T}\begin{bmatrix}\kappa_{1,t} - (\kappa_{1,t-1} + \theta_1) \\ \kappa_{2,t} - (\kappa_{2,t-1} - \theta_2)\end{bmatrix}\begin{bmatrix}\kappa_{1,t} - (\kappa_{1,t-1} + \theta_1) \\ \kappa_{2,t} - (\kappa_{2,t-1} - \theta_2)\end{bmatrix}'\right).
$$

Proof. To find the posterior of σ_k^2, let $\pi(\sigma_{kk}^2) \sim I.G(\frac{1}{2}, A_k)$ where $k \in (1,2)$ we then have the following,

$$\begin{aligned}
\pi(\sigma_1^2, \sigma_2^2|\Sigma_\omega) &\propto \pi(\Sigma_\omega|(\sigma_1^2, \sigma_2^2))\pi(\sigma_1^2, \sigma_2^2) \\
&\propto (\sigma_1^2\sigma_2^2)^{\frac{-(\nu+1)}{2}} \exp\left(-\nu\left[\frac{(\Sigma_\omega^{-1})_{11}}{\sigma_1^2} + \frac{(\Sigma_\omega^{-1})_{22}}{\sigma_2^2}\right]\right) \\
&\times(\sigma_1^2)^{-\frac{1}{2}-1} \exp\left(\frac{-1}{A_1\sigma_1^2}\right)(\sigma_2^2)^{-\frac{1}{2}-1} \exp\left(\frac{-1}{A_2\sigma_2^2}\right), \\
&\propto(\sigma_1^2)^{\frac{-(\nu+2)}{2}-1} \exp\left(-\left[\nu(\Sigma_\omega^{-1})_{11} + \frac{1}{A_1}\right]/\sigma_1^2\right) \\
&\times(\sigma_2^2)^{\frac{-(\nu+2)}{2}-1} \exp\left(-\left[\nu(\Sigma_\omega^{-1})_{22} + \frac{1}{A_2}\right]/\sigma_2^2\right).
\end{aligned}$$

We observe that the posteriors for σ_k^2 are independent of each other and that the conditional distribution has an Inverse Gamma kernel:

$$\pi(\sigma_k^2|\Sigma) \sim I.G\left(\frac{\nu+2}{2}, \nu(\Sigma_\omega^{-1})_{kk} + \frac{1}{A_k}\right) \text{ for } k \in (1,2) \tag{1.4}$$

The posterior for Σ can be found by,

$$\begin{aligned}
\pi(\Sigma|\boldsymbol{y_t}, \boldsymbol{\kappa_t}, \boldsymbol{\theta}) &\propto l(\boldsymbol{y}|\boldsymbol{\kappa_t}, \boldsymbol{\theta}, \Sigma)\pi(\boldsymbol{\kappa}|\boldsymbol{\theta}, \Sigma)\pi(\Sigma), \\
&\propto \pi(\boldsymbol{\kappa}|\boldsymbol{\theta}, \Sigma)\pi(\Sigma|\sigma_k^2) \quad \text{ for } k \in (1,2), \\
&\propto \prod_{t=1}^{T} \Sigma^{-\frac{1}{2}} \exp\left(-\frac{1}{2}[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]^T\Sigma^{-1}[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]\right) \\
&\times \exp\left(-\nu\left[\frac{(\Sigma_\omega^{-1})_{11}}{\sigma_1^2} + \frac{(\Sigma_\omega^{-1})_{22}}{\sigma_2^2}\right]\right) \\
&\propto \Sigma^{-\frac{T}{2}} \exp\sum_{t=1}^{T}\left(-\frac{1}{2}[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]^T\Sigma^{-1}[\boldsymbol{\kappa_t} - (\boldsymbol{\kappa_{t-1}} + \boldsymbol{\theta})]\right) \\
&\times|\Sigma| \exp\left(-\nu\left[\frac{(\Sigma_\omega^{-1})_{11}}{\sigma_1^2} + \frac{(\Sigma_\omega^{-1})_{22}}{\sigma_2^2}\right]\right).
\end{aligned}$$

□

Algorithm 4 EKF for CBD Binomial model

function EKF_$\kappa_t(\theta, \Sigma)$

Filter step

for $t = 1, ..., T$ **do**

Initiate:

$$a_{0|0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } P_{0|0} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}.$$

$$F_t = \begin{bmatrix} 1 & x_1 - \bar{x} \\ \vdots & \vdots \\ 1 & x_n - \bar{x} \end{bmatrix}$$

$$G_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Predictive:

$a_{t|t-1} = G_t a_{t-1|t-1} + \boldsymbol{\theta}$

$P_{t|t-1} = G_t P_{t-1|t-1} G_t' + \Sigma.$

Filter:

$\boldsymbol{\lambda_t} = F_t' a_{t|t-1}$, $\boldsymbol{\hat{Y}_t} = \frac{\mathrm{d}_t}{\mathrm{E}_t} \exp(-\boldsymbol{\lambda_t})(\iota_n + \exp(\boldsymbol{\lambda_t}))^2 - \exp(\boldsymbol{\lambda_t}) - \iota_n + \boldsymbol{\lambda_t}$

$\boldsymbol{\hat{V}_t} = \frac{1}{\mathrm{E}_t} \exp(\boldsymbol{\lambda_t})(\iota_n + \exp(\boldsymbol{\lambda_t}))^2$

$Q_t = F_t P_{t|t-1} F_t' + \mathrm{diag}(\boldsymbol{\hat{V}_t}),\ A_t = P_{t|t-1} F_t' Q_t^{-1}.$

$a_{t|t} = a_{t|t-1} + A_t \left(\boldsymbol{\hat{Y}_t} - \boldsymbol{\lambda_t} \right)$

$P_{t|t} = (\mathbb{I}_2 - A_t F_t)\, P_{t|t-1}.$

end for

Backward Smoothing step

for $t = T-1, ..., 1$ **do**

$h_t = a_{t|t} + P_{t|t} G_t P_{t|t-1}^{-1}(h_{t+1} - G_t a_{t+1|t+1})$

$H_t = P_{t|t} + P_{t|t} G_t P_{t|t-1}^{-1}(H_{t+1} - P_{t|t})(P_{t|t} G_t P_{t|t-1}^{-1})'.$

Sample $k_t \sim N(h_t, H_t)$.

end for

end function

Algorithm 5 MH for $\boldsymbol{\kappa_t}$ under CBD

function M-H_$\kappa_t(\boldsymbol{\kappa^*}, \boldsymbol{\kappa}, \boldsymbol{\theta}, \Sigma)$
  for $t = 1, ..., T$ **do**
    if $1 < t < T$ **then**
      $f^* = \sum_x(\mathrm{d}_{x,t}(\kappa^*_{1,t} + (x_i - \bar{x})\kappa^*_{2,t})) - \mathrm{E}_{x,t}(\log(1 + \exp(\kappa^*_{1,t} + (x_i - \bar{x})\kappa^*_{2,t})))$
      $-\frac{1}{2}(\boldsymbol{\kappa^*_t} - \boldsymbol{\theta})\Sigma^{-1}(\boldsymbol{\kappa^*_t} - \boldsymbol{\theta})' - \frac{1}{2}(\boldsymbol{\kappa^*_{t+1}} - \boldsymbol{\theta})\Sigma^{-1}(\boldsymbol{\kappa^*_{t+1}} - \boldsymbol{\theta})'$
      $f = \sum_x(\mathrm{d}_{x,t}(\kappa_{1,t} + (x_i - \bar{x})\kappa_{2,t}) - \mathrm{E}_{x,t}(\log(1 + \exp(\kappa_{1,t} + (x_i - \bar{x})\kappa_{2,t}))))$
      $-\frac{1}{2}(\boldsymbol{\kappa_t} - \boldsymbol{\theta})\Sigma^{-1}(\boldsymbol{\kappa_t} - \boldsymbol{\theta})' - \frac{1}{2}(\boldsymbol{\kappa_{t+1}} - \boldsymbol{\theta})\Sigma^{-1}(\boldsymbol{\kappa_{t+1}} - \boldsymbol{\theta})'$
      $\phi = \min(\exp(f^*/f), 1)$
    else
      $f^* = \sum_x(\mathrm{d}_{x,t}(\kappa^*_{1,t} + (x_i - \bar{x})\kappa^*_{2,t})) - \mathrm{E}_{x,t}(\log(1 + \exp(\kappa^*_{1,t} + (x_i - \bar{x})\kappa^*_{2,t}))$
      $-\frac{1}{2}(\boldsymbol{\kappa^*_t} - \boldsymbol{\theta})\Sigma^{-1}(\boldsymbol{\kappa^*_t} - \boldsymbol{\theta})'$
      $f = \sum_x(\mathrm{d}_{x,t}(\kappa_{1,t} + (x_i - \bar{x})\kappa_{2,t}) - \mathrm{E}_{x,t}(\log(1 + \exp(\kappa_{1,t} + (x_i - \bar{x})\kappa_{2,t}))))$
      $-\frac{1}{2}(\boldsymbol{\kappa_t} - \boldsymbol{\theta})\Sigma^{-1}(\boldsymbol{\kappa_t} - \boldsymbol{\theta})'$
      $\phi = \min(\exp(f^*/f), 1)$
    end if
    $u \sim U(0, 1)$
    if $u < \phi$ **then**
      $\hat{\boldsymbol{\kappa}}_t = \boldsymbol{\kappa^*_t}$
    else
      $\hat{\boldsymbol{\kappa}}_t = \boldsymbol{\kappa_t}$
    end if
  end for
end function

1.6. Bayesian model estimation

The model estimation was conducted for the following countries: Australia, United Kingdom, Italy, France, Spain, New-Zealand, Sweden, Germany, and Finland using the same prior distribution. We found similar convergence statistics for all countries using the trace-plot and Geweke Statistic.

Table 2 shows the posterior sample statistics for the static model parameters under the LC and CBD model respectively. To analyze the convergence of the chain we use the Geweke Statistic (Geweke et al., 1991) denoted by "Geweke" in our table. This diagnostic tool compares the last iterations from the MCMC samples to samples from the early iterations (after burn-in). If both sub-samples come from the same distribution then the Geweke Statistic is said to have a standard normal distribution. If the Geweke statistic is too large then we can reject the null hypothesis, implying that means of the two sub-samples are different, otherwise we will accept the null and say the chain has arrived at the stationary distribution. Observing the Geweke statistic, we see that in our case all static parameters have arrived at the stationary distribution.

Under the LC model structure, it is straightforward to see that α_x and ${}_{NL}\alpha_x$ will represent the average mortality rate of a person aged x over the time period T given the constraint that $\sum_{t=1}^{T} \kappa_t = 0$.

Figure 1 shows an increasing trend of α_x and ${}_{NL}\alpha_x$ across age. This is expected since after averaging the affects over time, we would expect a higher mortality rate for a higher age group. Under both the linear and nonlinear variants, we see that both α_x and ${}_{NL}\alpha_x$ have a similar trend and value. Observing Table 2, we see that α_{50}, and α_{90} not only have a similar value, but also a small posterior standard deviation as ${}_{NL}\alpha_{50}$ and ${}_{NL}\alpha_{90}$ and thus the tight posterior credible bounds. In the case of β_x and ${}_{NL}\beta_x$, it represents the age-specific changes in mortality rates over time, a general downward trend across ages demonstrates that there is an improvement in age-specific mortality rates across time. Figure 1 demonstrates that ${}_{NL}\beta_x$ seems to be a smoothed version of β_x and with a smaller variance. Combining the fact that $\boldsymbol{\alpha}$ represents the averaged (over time) mortality rate across age, and the constraint that $\sum_{x=x_1}^{x_n} \beta_x = 1$, we have

$$\sum_{x=x_1}^{x_n} (y_{x,t} - \overline{y_x}) = \kappa_t. \tag{1.5}$$

Here, $\overline{y_x}$ represents the average mortality rate from $t = 1$ to T, for a specific age x. From Eq. (1.5) we see that κ_t is the year to year changes in average mortality rates, and in general a decreasing trend in κ_t indicates that mortality rates over the years have improved. Figures 12 and 14 shows the fitted and forecasted κ_t and ${}_{NL}\kappa_t$ respectively, and similarly, they both show a downward trend, once again this hints at the apparent longevity risk issue. In this case the nonlinear LC model shows a larger variance, and this could be due to the numerical approximation from using the EKF and not enough simulations for the MH.

For the CBD model, it is driven by its two correlated hidden latent dynamics which drives the estimation of the death probability $q_{x,t}$. The first factor $\kappa_{1,t}$ controls the level of shift of mortality rates for a given year t for an individual aged x. For example a decreasing $\kappa_{1,t}$, will produce a parallel downward shift in mortality rates in a given year for an individual aged x, this will indicate that there exists mortality improvements over time. Whereas, for the second component, $\kappa_{t,2}$ represents the slope of mortality improvements across ages. For instance, if $\kappa_{t,2}$ were to have an upward trend, it would imply that mortality improvements for younger ages (those below $\bar{x}$) occurs more rapidly than older ages (those above $\bar{x}$). Figures 13 and 15 shows the fitted and forecasted latent dynamics of both the linear and nonlinear CBD model. The nonlinear CBD model demonstrates higher values for $\boldsymbol{\kappa_1}$ and $\boldsymbol{\kappa_2}$ compared with the linear CBD model, however, the trend is consistent between the two models. In essence, when Binomially distributed death counts are assumed, the resulting prediction for mortality rates tend to be higher in the nonlinear CBD model than if the crude mortality rates were used to fit the linear model.

1.6.1. Bayesian model: Static parameter analysis

Table 1: Hyper parameters for Measurement and State equations:

Linear model					
	Measurement parameters		State parameters		
LC	$\mu_\alpha = 0$	$\sigma^2_\alpha = 100$	$\mu_\theta = 0$	$\sigma^2_\theta = 100$	
	$\mu_\beta = 0$	$\sigma^2_\beta = 100$	$a_\omega = 2.1$	$b_\omega = 0.3$	
	$a_\varepsilon = 2.1$	$b_\varepsilon = 0.3$			
CBD	$a_\nu = 2.001$	$b_\nu = 0.001$	$\mu_\theta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\Sigma_\theta = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$	
			$A_1 = 1000$	$A_2 = 1000$	$\xi = 2$
Nonlinear model					
	Measurement parameters		State parameters		
LC	$\mu_\alpha = 0$	$\sigma^2_\alpha = 100$	$\mu_\theta = 0$	$\sigma^2_\theta = 100$	
	$\mu_\beta = 0$	$\sigma^2_\beta = 100$	$a_\omega = 2.1$	$b_\omega = 0.3$	
	$a_\alpha = 2.1$	$b_\alpha = 0.3$			
	a_β 2.1	$b_\beta = 0.3$			
CBD			$\mu_\theta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\Sigma_\theta = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$	
			$A_1 = 1000$	$A_2 = 1000$	$\xi = 2$

Table 2: Australian dataset: Bayesian statistics for linear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ_ε^2	2.8772×10^{-3}	$(2.7396 \times 10^{-3},\ 3.0196 \times 10^{-3})$	8.5923×10^{-4}	-1.1839
α_{50}	-5.0597	$(-5.0721,\ -5.0474)$	7.5033×10^{-3}	1.1827
α_{90}	-1.3770	$(-1.3896,\ -1.3646)$	7.5608×10^{-3}	-1.1749
β_{50}	3.1629×10^{-2}	$(3.0410 \times 10^{-2},\ 3.2849 \times 10^{-2})$	7.4058×10^{-4}	-1.3037
β_{90}	9.2071×10^{-3}	$(7.9936 \times 10^{-3}, 1.0436 \times 10^{-3})$	7.4249×10^{-4}	0.3632
δ	-6.2794×10^{-1}	$(-8.8651,\ -3.7127 \times 10^{-1})$	1.5791×10^{-1}	0.9006
σ_ω^2	1.2327	$(0.8251,\ 1.7563)$	2.9297×10^{-1}	-0.2431
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ_ν	4.2335×10^{-3}	$(3.9906 \times 10^{-3},\ 4.4887 \times 10^{-3})$	1.2746×10^{-4}	0.6584
θ_1	-1.435849×10^{-2}	$(-2.4114 \times 10^{-2},\ -4.8212 \times 10^{-3})$	4.9744×10^{-3}	0.7947
θ_2	4.1375×10^{-4}	$(4.7862 \times 10^{-5},\ 7.8433 \times 10^{-4})$	1.8505×10^{-4}	-0.1890
Σ_{11}	1.2691×10^{-3}	$(7.7430 \times 10^{-4},\ 2.0182 \times 10^{-3})$	3.2063×10^{-4}	-1.9144
Σ_{22}	1.7147×10^{-6}	$(8.2047 \times 10^{-7},\ 3.1110 \times 10^{-6})$	5.8713×10^{-7}	-1.0750
Σ_{12}	2.5153×10^{-5}	$(8.1556 \times 10^{-6},\ 4.8517 \times 10^{-5})$	1.0388×10^{-5}	0.2951

Table 3: Australian dataset: Bayesian statistics for nonlinear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-5.0570	$(-5.0675,\ -5.0465)$	6.3851×10^{-3}	1.6157
α_{90}	-1.3778	$(-1.3892,\ -1.3665)$	6.8575×10^{-3}	-1.2227
β_{50}	3.1532×10^{-2}	$(3.0212 \times 10^{-2},\ 3.2810 \times 10^{-2})$	5.8296×10^{-4}	1.0804
β_{90}	8.9609×10^{-3}	$(7.9933 \times 10^{-3}, 9.9162 \times 10^{-3})$	7.3514×10^{-4}	-0.3981
σ_β^2	8.7597×10^{-3}	$(2.4023 \times 10^{-3}, 2.2266 \times 10^{-2})$	1.8240×10^{-2}	0.3312
δ	-6.5183×10^{-1}	$(-9.3389 \times 10^{-1},\ -3.6904 \times 10^{-1})$	1.7185×10^{-1}	-0.7036
σ_ω^2	1.4562	$(1.0132,\ 2.0252)$	3.1526×10^{-1}	-0.8530
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	-1.5498×10^{-2}	$(-2.6258 \times 10^{-2}, -4.7760 \times 10^{-3})$	5.4754×10^{-3}	-0.0673
θ_2	3.7380×10^{-4}	$(-4.5737 \times 10^{-5},\ 7.8881 \times 10^{-4})$	2.1064×10^{-4}	-0.4029
Σ_{11}	1.5190×10^{-3}	$(9.9921 \times 10^{-4},\ 2.2778 \times 10^{-3})$	3.2797×10^{-4}	0.4929
Σ_{22}	2.2845×10^{-6}	$(1.3555 \times 10^{-6},\ 3.6601 \times 10^{-6})$	5.9516×10^{-7}	-0.1639
Σ_{12}	3.7998×10^{-5}	$(1.9822 \times 10^{-5},\ 6.3043 \times 10^{-5})$	1.1241×10^{-5}	0.2626

Table 4: United Kingdom dataset: Bayesian statistics for linear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ε	1.54324×10^{-3}	$(1.46779 \times 10^{-3}, 1.62166 \times 10^{-3})$	4.64433×10^{-5}	9.51935×10^{-1}
α_{50}	-5.26012	$(-5.26919, -5.25111)$	5.49700×10^{-3}	4.81996×10^{-1}
α_{90}	-1.47645	$(-1.48546, -1.46744)$	5.50912×10^{-3}	6.76301×10^{-1}
β_{50}	2.79223×10^{-2}	$(2.67873 \times 10^{-2}, 2.90668 \times 10^{-2})$	6.93065×10^{-4}	3.30651×10^{-1}
β_{90}	1.63317×10^{-2}	$(1.51903 \times 10^{-2}, 1.74473 \times 10^{-2})$	6.88232×10^{-4}	1.61227×10^{-1}
δ	-5.46649×10^{-1}	$(-8.40773 \times 10^{-1}, -2.52198 \times 10^{-1})$	1.79287×10^{-1}	1.51199
$\sigma^2\omega$	1.63142	$(1.11308, 2.29170)$	3.69547×10^{-1}	1.06908×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ν	1.05814×10^{-3}	$(9.98544 \times 10^{-4}, 1.12176 \times 10^{-3})$	3.13682×10^{-5}	-4.41183×10^{-1}
θ_1	-1.22802×10^{-2}	$(-2.18600 \times 10^{-2}, -2.92685 \times 10^{-3})$	4.83523×10^{-3}	-1.16150×10^{-1}
θ_2	1.41984×10^{-4}	$(-1.79677 \times 10^{-4}, 4.63129 \times 10^{-4})$	1.61731×10^{-4}	9.24107×10^{-1}
Σ_{11}	1.15420×10^{-3}	$(7.58485 \times 10^{-4}, 1.74192 \times 10^{-3})$	2.52342×10^{-4}	-6.81991×10^{-1}
Σ_{22}	1.30043×10^{-6}	$(7.93692 \times 10^{-7}, 2.04855 \times 10^{-6})$	3.24639×10^{-7}	6.66784×10^{-1}
Σ_{12}	2.88780×10^{-5}	$(1.67689 \times 10^{-5}, 4.60920 \times 10^{-5})$	7.58919×10^{-6}	-2.41716×10^{-1}

Table 5: United Kingdom dataset: Bayesian statistics for nonlinear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-5.25845	$(-5.26231, -5.25455)$	2.36065×10^{-3}	-1.51772
α_{90}	-1.47942	$(-1.48197, -1.47687)$	1.55920×10^{-3}	3.67529×10^{-1}
β_{50}	2.80215×10^{-2}	$(2.75185 \times 10^{-2}, 2.85112 \times 10^{-2})$	2.98838×10^{-4}	1.96526
β_{90}	1.57908×10^{-2}	$(1.54704 \times 10^{-2}, 1.61048 \times 10^{-2})$	1.92163×10^{-4}	-5.72042×10^{-1}
σ^2_β	7.60858×10^{-3}	$(2.14411 \times 10^{-3}, 1.96781 \times 10^{-2})$	1.12651×10^{-2}	6.01330×10^{-1}
δ	-5.33103×10^{-1}	$(-8.82784 \times 10^{-1}, -1.88404 \times 10^{-1})$	2.10745×10^{-1}	-4.40864×10^{-1}
σ^2_ω	2.23694	$(1.59761, 3.07974)$	4.59908×10^{-1}	4.54231×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	-1.28944×10^{-2}	$(-2.32472 \times 10^{-2}, -2.71482 \times 10^{-3})$	5.18756×10^{-3}	-1.98287
θ_2	1.12437×10^{-4}	$(-3.21968 \times 10^{-4}, 5.41054 \times 10^{-4})$	2.19517×10^{-4}	8.19264×10^{-1}
Σ_{11}	1.33976×10^{-3}	$(9.04994 \times 10^{-4}, 1.96933 \times 10^{-3})$	2.75550×10^{-4}	-1.39999
Σ_{22}	2.38100×10^{-6}	$(1.58462 \times 10^{-6}, 3.56225 \times 10^{-6})$	5.07394×10^{-7}	-1.32071
Σ_{12}	4.07204×10^{-5}	$(2.45055 \times 10^{-5}, 6.40629 \times 10^{-5})$	1.00837×10^{-5}	-1.47180

Table 6: Italy dataset: Bayesian statistics for linear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ε	1.14676×10^{-3}	$(1.09110 \times 10^{-3}, 1.20438 \times 10^{-3})$	3.46237×10^{-5}	8.12461×10^{-1}
α_{50}	-5.34170	$(-5.34952, -5.33396)$	4.74696×10^{-3}	1.02683
α_{90}	-1.42623	$(-1.43408 , -1.41841)$	4.78464×10^{-3}	4.79464×10^{-1}
β_{50}	2.63627×10^{-2}	$(2.55838 \times 10^{-2}, 2.71470 \times 10^{-2})$	4.75330×10^{-4}	2.15852×10^{-1}
β_{90}	1.68317×10^{-2}	$(1.60467 \times 10^{-2}, 1.75965 \times 10^{-2})$	4.68859×10^{-4}	-3.64945×10^{-1}
δ	-6.26693×10^{-1}	$(-1.03160, -2.21561 \times 10^{-1})$	2.48065×10^{-1}	1.06457
$\sigma^2\omega$	3.09973	$(2.19022, 4.29212)$	6.57897×10^{-1}	1.44059×10^{-2}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ν	2.84213×10^{-3}	$(2.68091 \times 10^{-3}, 3.01077 \times 10^{-3})$	8.47068×10^{-5}	9.97722×10^{-1}
θ_1	-1.38571×10^{-2}	$(-2.58245 \times 10^{-2}, -1.81424 \times 10^{-3})$	6.11089×10^{-3}	-1.44482
θ_2	1.32924×10^{-4}	$(-2.03764 \times 10^{-4}, 4.67026 \times 10^{-4})$	1.69946×10^{-4}	-1.12748
Σ_{11}	1.85646×10^{-3}	$(1.20046 \times 10^{-3}, 2.83510 \times 10^{-3})$	4.13812×10^{-4}	-6.77242×10^{-1}
Σ_{22}	1.44079×10^{-6}	$(7.49236 \times 10^{-7}, 2.50423 \times 10^{-6})$	4.56908×10^{-7}	-3.71208×10^{-1}
Σ_{12}	3.89637×10^{-5}	$(2.11584 \times 10^{-5}, 6.52088 \times 10^{-5})$	1.12705×10^{-5}	4.99075×10^{-1}

Table 7: Italy dataset: Bayesian statistics for nonlinear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-5.34091	$(-5.34500, -5.33676)$	2.49264×10^{-3}	-1.00539
α_{90}	-1.42690	$(-1.42987, -1.42394)$	1.80751×10^{-3}	1.40962
β_{50}	2.64204×10^{-2}	$(2.59869 \times 10^{-2}, 2.68469 \times 10^{-2})$	2.62386×10^{-4}	1.20669
β_{90}	1.65914×10^{-2}	$(1.63214 \times 10^{-2}, 1.68581 \times 10^{-2})$	1.61967×10^{-4}	4.80742×10^{-1}
σ^2_β	7.41985×10^{-3}	$(2.10497 \times 10^{-3}, 1.94495 \times 10^{-2})$	1.00503×10^{-2}	8.11224×10^{-1}
δ	-6.10723×10^{-1}	$(-1.04564, -1.82459 \times 10^{-1})$	2.63049×10^{-1}	-6.92291×10^{-1}
σ^2_ω	3.46360	$(2.48998, 4.75233)$	7.02411×10^{-1}	2.11812×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	-1.47919×10^{-2}	$(-2.85549 \times 10^{-2}, -1.36433 \times 10^{-3})$	6.93802×10^{-3}	-9.76333×10^{-1}
θ_2	1.46284×10^{-4}	$(-4.27702 \times 10^{-4}, 7.36595 \times 10^{-4})$	2.95392×10^{-4}	2.54163×10^{-1}
Σ_{11}	2.36693×10^{-3}	$(1.59237 \times 10^{-3}, 3.51196 \times 10^{-3})$	4.93707×10^{-4}	-2.69993
Σ_{22}	4.24591×10^{-6}	$(2.84328 \times 10^{-6}, 6.31977 \times 10^{-6})$	8.93119×10^{-7}	-6.81762×10^{-1}
Σ_{12}	6.92716×10^{-5}	$(4.12404 \times 10^{-5}, 1.09812 \times 10^{-4})$	1.77002×10^{-5}	-1.80021

Table 8: France dataset: Bayesian statistics for linear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ε	9.81144×10^{-4}	$(9.33472 \times 10^{-4}, 1.03038 \times 10^{-3})$	2.95542×10^{-5}	7.79589×10^{-1}
α_{50}	-5.11901	$(-5.12623, -5.11184)$	4.38111×10^{-3}	1.02157
α_{90}	-1.48864	$(-1.49587, -1.48143)$	4.40246×10^{-3}	4.77981×10^{-1}
β_{50}	2.01200×10^{-2}	$(1.94361 \times 10^{-2}, 2.08104 \times 10^{-2})$	4.17425×10^{-4}	2.10995×10^{-1}
β_{90}	1.68320×10^{-2}	$(1.61467 \times 10^{-2}, 1.75050 \times 10^{-2})$	4.12868×10^{-4}	-3.34664×10^{-1}
δ	-6.60950×10^{-1}	$(-1.02119, -3.00274 \times 10^{-1})$	2.20336×10^{-1}	1.07581
$\sigma^2\omega$	2.46058	$(1.73077, 3.40319)$	5.23755×10^{-1}	1.98285×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ν	1.02347×10^{-2}	$(9.64907 \times 10^{-3}, 1.08576 \times 10^{-2})$	3.06174×10^{-4}	-2.38777×10^{-1}
θ_1	-1.48080×10^{-2}	$(-2.22286 \times 10^{-2}, -7.48229 \times 10^{-3})$	3.70322×10^{-3}	-2.37797×10^{-1}
θ_2	4.27777×10^{-5}	$(-3.06746 \times 10^{-5}, 1.27425 \times 10^{-4})$	3.91728×10^{-5}	1.26138
Σ_{11}	6.87106×10^{-4}	$(2.90385 \times 10^{-4}, 1.30376 \times 10^{-3})$	2.63726×10^{-4}	4.01244×10^{-1}
Σ_{22}	6.24264×10^{-8}	$(3.46109 \times 10^{-9}, 2.62899 \times 10^{-7})$	7.61815×10^{-8}	4.65728×10^{-3}
Σ_{12}	6.34534×10^{-7}	$(-4.23506 \times 10^{-6}, 8.75130 \times 10^{-6})$	3.20954×10^{-6}	-2.12098×10^{-2}

Table 9: France dataset: Bayesian statistics for nonlinear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-5.11627	$(-5.12010, -5.11239)$	2.33938×10^{-3}	-8.26416×10^{-1}
α_{90}	-1.48945	$(-1.49209, -1.48685)$	1.59271×10^{-3}	7.94091×10^{-1}
β_{50}	2.03660×10^{-2}	$(2.00110 \times 10^{-2}, 2.07240 \times 10^{-2})$	2.18445×10^{-4}	6.18083×10^{-1}
β_{90}	1.66348×10^{-2}	$(1.63909 \times 10^{-2}, 1.68739 \times 10^{-2})$	1.43997×10^{-4}	-5.42090×10^{-1}
σ^2_β	7.53469×10^{-3}	$(2.11820 \times 10^{-3}, 1.94962 \times 10^{-2})$	1.11942×10^{-2}	5.85613×10^{-1}
δ	-6.53380×10^{-1}	$(-1.03716, -2.77048 \times 10^{-1})$	2.30651×10^{-1}	-6.89746×10^{-1}
σ^2_ω	2.67919	$(1.92092, 3.68848)$	5.47945×10^{-1}	1.03989×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	-1.58804×10^{-2}	$(-2.72846 \times 10^{-2}, -4.31451 \times 10^{-3})$	5.87076×10^{-3}	-1.22015
θ_2	1.32616×10^{-4}	$(-3.05353 \times 10^{-4}, 5.78872 \times 10^{-4})$	2.23582×10^{-4}	-7.32388×10^{-1}
Σ_{11}	1.72326×10^{-3}	$(1.15549 \times 10^{-3}, 2.54363 \times 10^{-3})$	3.58734×10^{-4}	6.26519×10^{-2}
Σ_{22}	2.51982×10^{-6}	$(1.66904 \times 10^{-6}, 3.74754 \times 10^{-6})$	5.38562×10^{-7}	-1.10957×10^{-2}
Σ_{12}	4.63384×10^{-5}	$(2.73664 \times 10^{-5}, 7.23309 \times 10^{-5})$	1.16994×10^{-5}	-2.86474×10^{-1}

Table 10: Spain dataset: Bayesian statistics for linear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ε	9.81144×10^{-4}	$(9.33472 \times 10^{-4}, 1.03038 \times 10^{-3})$	2.95542×10^{-5}	7.79589×10^{-1}
α_{50}	-5.11901	$(-5.12623, -5.11184)$	4.38111×10^{-3}	1.02157
α_{90}	-1.48864	$(-1.49587, -1.48143)$	4.40246×10^{-3}	4.77981×10^{-1}
β_{50}	2.01200×10^{-2}	$(1.94361 \times 10^{-2}, 2.08104 \times 10^{-2})$	4.17425×10^{-4}	2.10995×10^{-1}
β_{90}	1.68320×10^{-2}	$(1.61467 \times 10^{-2}, 1.75050 \times 10^{-2})$	4.12868×10^{-4}	-3.34664×10^{-1}
δ	-6.60950×10^{-1}	$(-1.02119, -3.00274 \times 10^{-1})$	2.20336×10^{-1}	1.07581
$\sigma^2\omega$	2.46058	$(1.73077, 3.40319)$	5.23755×10^{-1}	1.98285×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ν	1.02347×10^{-2}	$(9.64907 \times 10^{-3}, 1.08576 \times 10^{-2})$	3.06174×10^{-4}	-2.38777×10^{-1}
θ_1	-1.48080×10^{-2}	$(-2.22286 \times 10^{-2}, -7.48229 \times 10^{-3})$	3.70322×10^{-3}	-2.37797×10^{-1}
θ_2	4.27777×10^{-5}	$(-3.06746 \times 10^{-5}, 1.27425 \times 10^{-4})$	3.91728×10^{-5}	1.26138
Σ_{11}	6.87106×10^{-4}	$(2.90385 \times 10^{-4}, 1.30376 \times 10^{-3})$	2.63726×10^{-4}	4.01244×10^{-1}
Σ_{22}	6.24264×10^{-8}	$(3.46109 \times 10^{-9}, 2.62899 \times 10^{-7})$	7.61815×10^{-8}	4.65728×10^{-3}
Σ_{12}	6.34534×10^{-7}	$(-4.23506 \times 10^{-6}, 8.75130 \times 10^{-6})$	3.20954×10^{-6}	-2.12098×10^{-2}

Table 11: Spain dataset: Bayesian statistics for nonlinear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-5.32868	$(-5.33401, -5.32341)$	3.21159×10^{-3}	-7.38557×10^{-1}
α_{90}	-1.46963	$(-1.47350, -1.46566)$	2.38296×10^{-3}	9.93451×10^{-1}
β_{50}	2.70877×10^{-2}	$(2.65830 \times 10^{-2}, 2.75868 \times 10^{-2})$	3.06601×10^{-4}	-9.57635×10^{-1}
β_{90}	1.45684×10^{-2}	$(1.42231 \times 10^{-2}, 1.49062 \times 10^{-2})$	2.07411×10^{-4}	2.26166×10^{-1}
σ^2_β	8.03898×10^{-3}	$(2.28841 \times 10^{-3}, 2.08555 \times 10^{-2})$	1.05372×10^{-2}	9.51696×10^{-1}
δ	-6.45177×10^{-1}	$(-1.13815, -1.55218 \times 10^{-1})$	3.01437×10^{-1}	-1.35080×10^{-2}
σ^2_ω	4.49362	$(3.21016, 6.17900)$	9.25570×10^{-1}	1.81127
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	-1.61953×10^{-2}	$(-3.22370 \times 10^{-2}, -1.21837 \times 10^{-4})$	8.11926×10^{-3}	3.17641×10^{-2}
θ_2	3.45084×10^{-4}	$(-2.58143 \times 10^{-5}, 7.22063 \times 10^{-4})$	1.89760×10^{-4}	-2.74123
Σ_{11}	3.33126×10^{-3}	$(2.22919 \times 10^{-3}, 4.98537 \times 10^{-3})$	7.01493×10^{-4}	1.56459×10^{-1}
Σ_{22}	1.78937×10^{-6}	$(1.15788 \times 10^{-6}, 2.72878 \times 10^{-6})$	4.01880×10^{-7}	-1.56441×10^{-1}
Σ_{12}	4.89689×10^{-5}	$(2.68424 \times 10^{-5}, 8.02832 \times 10^{-5})$	1.36767×10^{-5}	-7.59272×10^{-1}

Table 12: New Zealand dataset: Bayesian statistics for linear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ε	5.31393×10^{-3}	$(5.05644 \times 10^{-3}, 5.57745 \times 10^{-3})$	1.59569×10^{-4}	7.31028×10^{-1}
α_{50}	-5.25626	$(-5.27293, -5.23951)$	1.01855×10^{-2}	4.66605×10^{-1}
α_{90}	-1.53411	$(-1.55106, -1.51704)$	1.03443×10^{-2}	5.06526×10^{-1}
β_{50}	2.78461×10^{-2}	$(2.57100 \times 10^{-2}, 2.99883 \times 10^{-2})$	1.29104×10^{-3}	-8.20311×10^{-2}
β_{90}	1.34209×10^{-2}	$(1.12902 \times 10^{-2}, 1.55657 \times 10^{-2})$	1.28875×10^{-3}	-2.05925
δ	-5.39540×10^{-1}	$(-7.91956 \times 10^{-1}, -2.90266 \times 10^{-1})$	1.52595×10^{-1}	-1.12914
$\sigma^2\omega$	1.15155	$(7.28017 \times 10^{-1}, 1.71738)$	3.06452×10^{-1}	-4.64180×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ^2_ν	6.23817×10^{-3}	$(5.88368 \times 10^{-3}, 6.61510 \times 10^{-3})$	1.85146×10^{-4}	-7.78508×10^{-1}
θ_1	-1.26664×10^{-2}	$(-2.12938 \times 10^{-2}, -4.14963 \times 10^{-3})$	4.34135×10^{-3}	2.72185
θ_2	1.47823×10^{-4}	$(-2.96699 \times 10^{-4}, 5.93029 \times 10^{-4})$	2.22596×10^{-4}	1.86718
Σ_{11}	9.34472×10^{-4}	$(5.39661 \times 10^{-4}, 1.51642 \times 10^{-3})$	2.51303×10^{-4}	1.66998×10^{-1}
Σ_{22}	2.49912×10^{-6}	$(1.21007 \times 10^{-6}, 4.50818 \times 10^{-6})$	8.49604×10^{-7}	6.76799×10^{-1}
Σ_{12}	2.64816×10^{-5}	$(9.23150 \times 10^{-6}, 5.04334 \times 10^{-5})$	1.05695×10^{-5}	5.26959×10^{-1}

Table 13: New Zealand dataset: Bayesian statistics for nonlinear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-5.25168	$(-5.27013, -5.23332)$	1.11521×10^{-2}	5.08668×10^{-1}
α_{90}	-1.53417	$(-1.54787, -1.52040)$	8.33181×10^{-3}	6.23026×10^{-1}
β_{50}	2.69265×10^{-2}	$(2.34758 \times 10^{-2}, 2.98256 \times 10^{-2})$	1.93819×10^{-3}	1.70159
β_{90}	1.33900×10^{-2}	$(1.17795 \times 10^{-2}, 1.49606 \times 10^{-2})$	9.81728×10^{-4}	9.46450×10^{-1}
σ^2_β	7.68609×10^{-3}	$(2.16828 \times 10^{-3}, 1.93964 \times 10^{-2})$	1.18563×10^{-2}	3.55217×10^{-1}
δ	-5.46115×10^{-1}	$(-8.43323 \times 10^{-1}, -2.53381 \times 10^{-1})$	1.80397×10^{-1}	1.59851
σ^2_ω	1.63320	$(1.10136, 2.32012)$	3.80174×10^{-1}	-2.26953
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	-1.33755×10^{-2}	$(-2.30690 \times 10^{-2}, -3.53557 \times 10^{-3})$	4.97902×10^{-3}	1.07265
θ_2	1.38617×10^{-4}	$(-3.65047 \times 10^{-4}, 6.41555 \times 10^{-4})$	2.55198×10^{-4}	3.93530×10^{-3}
Σ_{11}	1.22190×10^{-3}	$(7.76689 \times 10^{-4}, 1.86674 \times 10^{-3})$	2.85039×10^{-4}	-1.63068×10^{-1}
Σ_{22}	3.23604×10^{-6}	$(1.81781 \times 10^{-6}, 5.32995 \times 10^{-6})$	9.07358×10^{-7}	-8.30731×10^{-1}
Σ_{12}	4.18911×10^{-5}	$(2.16849 \times 10^{-5}, 7.09314 \times 10^{-5})$	1.26986×10^{-5}	-1.39923

Table 14: Sweden dataset: Bayesian statistics for linear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ_ε^2	2.10347×10^{-3}	$(1.99682 \times 10^{-3}, 2.21463 \times 10^{-3})$	6.60748×10^{-5}	-9.41591×10^{-1}
α_{50}	-5.50930	$(-5.51994, -5.49884)$	6.39662×10^{-3}	-2.09066×10^{-1}
α_{90}	-1.46814	$(-1.47873, -1.45752)$	6.44443×10^{-3}	2.32321
β_{50}	2.11893×10^{-2}	$(1.98811 \times 10^{-2}, 2.25104 \times 10^{-2})$	7.97686×10^{-4}	-7.54657×10^{-1}
β_{90}	1.69868×10^{-2}	$(1.56712 \times 10^{-2}, 1.83124 \times 10^{-2})$	8.00541×10^{-4}	4.98373×10^{-1}
δ	-5.44697×10^{-1}	$(-7.29142 \times 10^{-1}, -3.61351 \times 10^{-1})$	1.11662×10^{-1}	-1.17708
$\sigma^2\omega$	6.13857×10^{-1}	$(3.84508 \times 10^{-1}, 9.18754 \times 10^{-1})$	1.65866×10^{-1}	1.79522
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ_ν^2	3.19388×10^{-3}	$(3.01597 \times 10^{-3}, 3.38522 \times 10^{-3})$	9.41953×10^{-5}	5.11276×10^{-1}
θ_1	-1.26949×10^{-2}	$(-1.80632 \times 10^{-2}, -7.40684 \times 10^{-3})$	2.69201×10^{-3}	1.61812×10^{-1}
θ_2	6.54282×10^{-5}	$(-1.89294 \times 10^{-4}, 3.16370 \times 10^{-4})$	1.27749×10^{-4}	9.87482×10^{-1}
Σ_{11}	3.55700×10^{-4}	$(1.84609 \times 10^{-4}, 6.08669 \times 10^{-4})$	1.09897×10^{-4}	4.12607×10^{-1}
Σ_{22}	8.12024×10^{-7}	$(3.85535 \times 10^{-7}, 1.48904 \times 10^{-6})$	2.85006×10^{-7}	-2.22987×10^{-1}
Σ_{12}	9.28391×10^{-6}	$(2.31510 \times 10^{-6}, 1.88306 \times 10^{-5})$	4.19521×10^{-6}	8.46213×10^{-2}

Table 15: Sweden dataset: Bayesian statistics for nonlinear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-5.50927	$(-5.52051, -5.49787)$	6.87182×10^{-3}	-3.06384×10^{-1}
α_{90}	-1.47077	$(-1.47726, -1.46426)$	3.94732×10^{-3}	-1.06600
β_{50}	2.13392×10^{-2}	$(1.98385 \times 10^{-2}, 2.27459 \times 10^{-2})$	8.86165×10^{-4}	-5.54941×10^{-1}
β_{90}	1.66185×10^{-2}	$(1.58377 \times 10^{-2}, 1.73921 \times 10^{-2})$	4.73708×10^{-4}	-2.22508
σ_β^2	7.54000×10^{-3}	$(2.12666 \times 10^{-3}, 1.92063 \times 10^{-2})$	1.30136×10^{-2}	7.19696×10^{-1}
δ	-5.59174×10^{-1}	$(-7.85572 \times 10^{-1}, -3.35755 \times 10^{-1})$	1.37801×10^{-1}	2.05958
σ_ω^2	9.41917×10^{-1}	$(6.44728 \times 10^{-1}, 1.33041)$	2.12606×10^{-1}	9.86550×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	-1.38611×10^{-2}	$(-2.03532 \times 10^{-2}, -7.26658 \times 10^{-3})$	3.30723×10^{-3}	-1.86891
θ_2	8.12012×10^{-5}	$(-2.54967 \times 10^{-4}, 4.14612 \times 10^{-4})$	1.69752×10^{-4}	1.37585×10^{-1}
Σ_{11}	5.53216×10^{-4}	$(3.59058 \times 10^{-4}, 8.39161 \times 10^{-4})$	1.23030×10^{-4}	1.06571
Σ_{22}	1.42000×10^{-6}	$(8.47160 \times 10^{-7}, 2.27927 \times 10^{-6})$	3.66915×10^{-7}	4.49116×10^{-1}
Σ_{12}	1.79952×10^{-5}	$(9.37834 \times 10^{-6}, 3.00097 \times 10^{-5})$	5.28502×10^{-6}	2.32130

Table 16: Germany dataset: Bayesian statistics for linear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ_ε^2	2.64788×10^{-3}	$(2.31988 \times 10^{-3}, 3.02503 \times 10^{-3})$	2.14131×10^{-4}	1.44626
α_{50}	-5.44934	$(-5.47487, -5.42384)$	1.54922×10^{-2}	1.67552×10^{-1}
α_{90}	-1.55641	$(-1.58248, -1.53075)$	1.56315×10^{-2}	1.00726
β_{50}	2.19152×10^{-2}	$(1.39835 \times 10^{-2}, 2.98302 \times 10^{-2})$	4.82912×10^{-3}	-2.00529
β_{90}	1.69333×10^{-2}	$(8.97917 \times 10^{-3}, 2.48504 \times 10^{-2})$	4.84691×10^{-3}	-1.32163
δ	-7.62263×10^{-1}	$(-1.04226, -5.19928 \times 10^{-1})$	1.61397×10^{-1}	4.47497×10^{-1}
$\sigma^2\omega$	2.03682×10^{-1}	$(7.64998 \times 10^{-2}, 4.27917 \times 10^{-1})$	1.21150×10^{-1}	-2.69823
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ_ν^2	7.87532×10^{-3}	$(6.73054 \times 10^{-3}, 9.57367 \times 10^{-3})$	7.23887×10^{-4}	-1.54879
θ_1	-1.88710×10^{-2}	$(-2.87000 \times 10^{-2}, -1.04939 \times 10^{-2})$	4.59382×10^{-3}	-8.98424×10^{-1}
θ_2	1.31071×10^{-4}	$(-3.38709 \times 10^{-4}, 6.05844 \times 10^{-4})$	2.36942×10^{-4}	-1.30797
Σ_{11}	1.34891×10^{-4}	$(2.02555 \times 10^{-5}, 5.00471 \times 10^{-4})$	1.47573×10^{-4}	8.33307×10^{-1}
Σ_{22}	3.86475×10^{-7}	$(2.07930 \times 10^{-8}, 1.71527 \times 10^{-6})$	5.38736×10^{-7}	-6.50306×10^{-1}
Σ_{12}	-3.79812×10^{-7}	$(-7.14822 \times 10^{-6}, 6.09360 \times 10^{-6})$	3.23584×10^{-6}	4.01511×10^{-1}

Table 17: Germany dataset: Bayesian statistics for nonlinear model static parameters

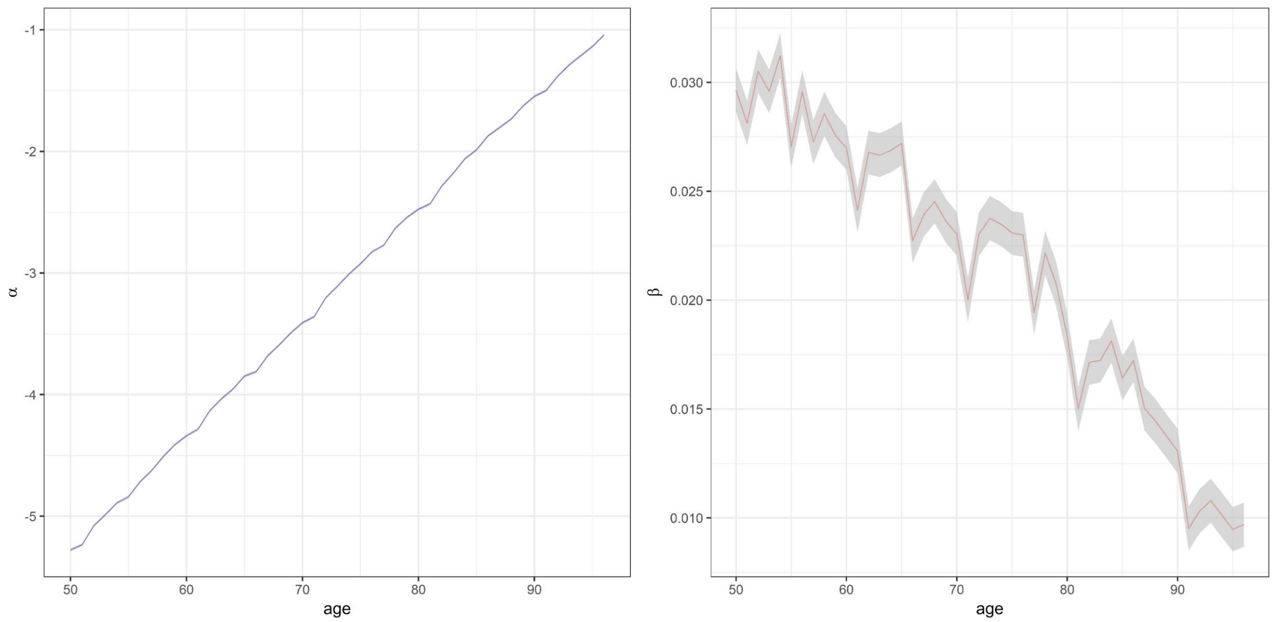
LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-5.45017	$(-5.45775, -5.44272)$	4.57306×10^{-3}	7.87967×10^{-2}
α_{90}	-1.55638	$(-1.55979, -1.55297)$	2.06864×10^{-3}	1.37232
β_{50}	2.06352×10^{-2}	$(1.84110 \times 10^{-2}, 2.29934 \times 10^{-2})$	1.40758×10^{-3}	1.43421
β_{90}	1.66984×10^{-2}	$(1.56020 \times 10^{-2}, 1.78570 \times 10^{-2})$	6.86063×10^{-4}	1.68999
σ_β^2	7.85116×10^{-3}	$(2.21954 \times 10^{-3}, 2.02117 \times 10^{-2})$	1.07671×10^{-2}	7.34894×10^{-1}
δ	-1.03781	$(-1.39502, -6.79406 \times 10^{-1})$	2.20544×10^{-1}	6.62579×10^{-1}
σ_ω^2	4.87103×10^{-1}	$(2.00327 \times 10^{-1}, 9.72396 \times 10^{-1})$	2.66098×10^{-1}	-2.24820
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	-2.50802×10^{-2}	$(-3.42011 \times 10^{-2}, -1.58488 \times 10^{-2})$	4.61784×10^{-3}	-8.05455×10^{-2}
θ_2	1.57626×10^{-4}	$(-3.46527 \times 10^{-4}, 6.49549 \times 10^{-4})$	2.49043×10^{-4}	-6.03623×10^{-1}
Σ_{11}	2.09140×10^{-4}	$(7.52619 \times 10^{-5}, 5.43726 \times 10^{-4})$	1.44395×10^{-4}	2.63237×10^{-2}
Σ_{22}	6.17707×10^{-7}	$(2.17810 \times 10^{-7}, 1.60303 \times 10^{-6})$	3.81275×10^{-7}	-1.49260
Σ_{12}	3.79773×10^{-6}	$(-2.60516 \times 10^{-6}, 1.35096 \times 10^{-5})$	4.14637×10^{-6}	-2.09064×10^{-1}

Table 18: Russia dataset: Bayesian statistics for linear model static parameters

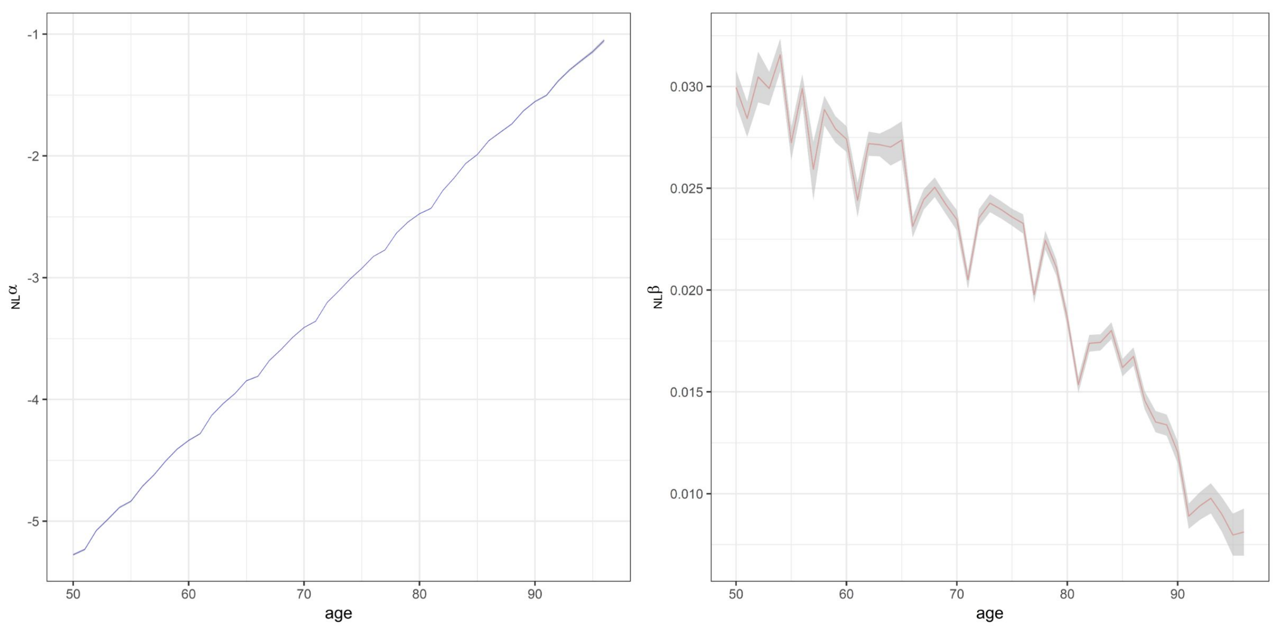
LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ_ε^2	3.84529×10^{-3}	$(3.64045 \times 10^{-3}, 4.05663 \times 10^{-3})$	1.27228×10^{-4}	-5.06831×10^{-1}
α_{50}	-4.64205	$(-4.65767, -4.62641)$	9.55595×10^{-3}	-6.75382×10^{-2}
α_{90}	-1.42108	$(-1.43684, -1.40527)$	9.54301×10^{-3}	-3.88792×10^{-1}
β_{50}	4.29081×10^{-2}	$(3.98923 \times 10^{-2}, 4.59308 \times 10^{-2})$	1.82932×10^{-3}	-5.68346×10^{-1}
β_{90}	7.29256×10^{-3}	$(4.31405 \times 10^{-3}, 1.02749 \times 10^{-2})$	1.80369×10^{-3}	-2.14750×10^{-1}
δ	3.79509×10^{-1}	$(-2.54567 \times 10^{-2}, 7.89295 \times 10^{-1})$	2.47676×10^{-1}	-3.61402×10^{-1}
$\sigma^2\omega$	2.50625	$(1.68332, 3.62445)$	6.06158×10^{-1}	3.34020×10^{-2}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
σ_ν^2	1.30523×10^{-2}	$(1.22498 \times 10^{-2}, 1.39175 \times 10^{-2})$	4.22285×10^{-4}	-9.83081×10^{-1}
θ_1	8.78641×10^{-3}	$(-3.81514 \times 10^{-3}, 2.13244 \times 10^{-2})$	6.35929×10^{-3}	-3.25052×10^{-1}
θ_2	-3.53500×10^{-5}	$(-4.78665 \times 10^{-4}, 4.10329 \times 10^{-4})$	2.26183×10^{-4}	1.96351×10^{-2}
Σ_{11}	1.63027×10^{-3}	$(9.16806 \times 10^{-4}, 2.78488 \times 10^{-3})$	4.85648×10^{-4}	-1.73321×10^{-1}
Σ_{22}	2.07219×10^{-6}	$(1.02521 \times 10^{-6}, 3.82687 \times 10^{-6})$	7.23320×10^{-7}	2.79564×10^{-2}
Σ_{12}	-3.20773×10^{-5}	$(-6.29819 \times 10^{-5}, -7.45585 \times 10^{-6})$	1.40439×10^{-5}	-3.11206×10^{-1}

Table 19: Russia dataset: Bayesian statistics for nonlinear model static parameters

LC model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
α_{50}	-4.64428	$(-4.64638, -4.64220)$	1.27629×10^{-3}	-2.50764×10^{-1}
α_{90}	-1.42287	$(-1.42528, -1.42043)$	1.46327×10^{-3}	4.82918×10^{-1}
β_{50}	4.45935×10^{-2}	$(4.41963 \times 10^{-2}, 4.50090 \times 10^{-2})$	2.46863×10^{-4}	4.90581×10^{-1}
β_{90}	9.16044×10^{-3}	$(8.74849 \times 10^{-3}, 9.56530 \times 10^{-3})$	2.49662×10^{-4}	5.27157×10^{-1}
σ_β^2	9.65127×10^{-3}	$(2.70022 \times 10^{-3}, 2.47764 \times 10^{-2})$	1.32496×10^{-2}	6.36979×10^{-1}
δ	3.52241×10^{-1}	$(-9.18942 \times 10^{-2}, 7.91125 \times 10^{-1})$	2.69821×10^{-1}	2.89638×10^{-1}
σ_ω^2	2.96163	$(2.06568, 4.17071)$	6.56461×10^{-1}	-6.95620×10^{-1}
CBD model				
Parameter	Posterior Mean	95% Credible Interval	Posterior St.Dev	Geweke
θ_1	6.36255×10^{-3}	$(-8.02227 \times 10^{-3}, 2.07710 \times 10^{-2})$	7.26635×10^{-3}	-8.12630×10^{-1}
θ_2	-2.06571×10^{-4}	$(-8.33714 \times 10^{-4}, 4.17240 \times 10^{-4})$	3.17594×10^{-4}	-6.47511×10^{-1}
Σ_{11}	2.17851×10^{-3}	$(1.38105 \times 10^{-3}, 3.39793 \times 10^{-3})$	5.19287×10^{-4}	1.50144×10^{-1}
Σ_{22}	4.10205×10^{-6}	$(2.60073 \times 10^{-6}, 6.42891 \times 10^{-6})$	9.82761×10^{-7}	2.70250×10^{-1}
Σ_{12}	-6.06009×10^{-6}	$(-3.70908 \times 10^{-5}, 2.34584 \times 10^{-5})$	1.51686×10^{-5}	1.00780

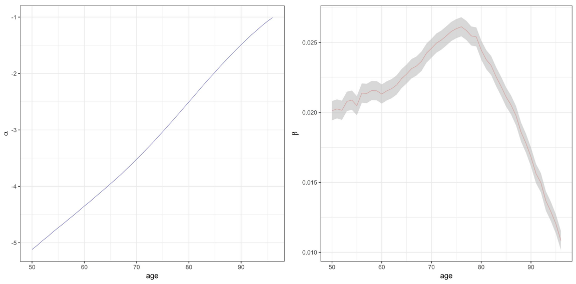


(a) Linear Estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

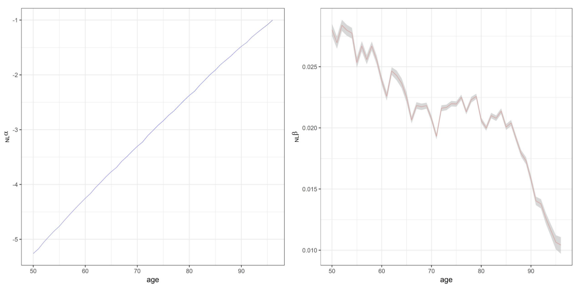


(b) Non-Linear Estimation of ${}_{\boldsymbol{NL}}\boldsymbol{\alpha}$ and ${}_{\boldsymbol{NL}}\boldsymbol{\beta}$.

Figure 1: 95% HPD of α and β under the linear and nonlinear LC Model for the Australian dataset.

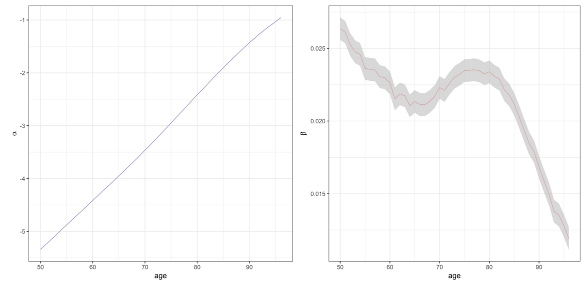


Linear estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

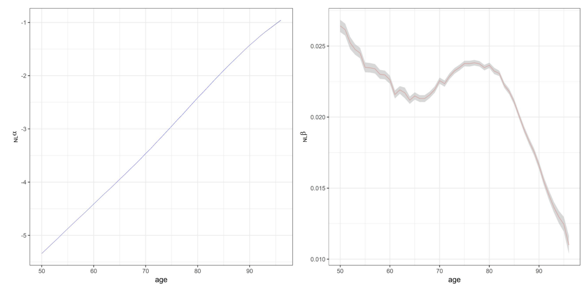


Nonlinear estimation of ${}_{NL}\boldsymbol{\alpha}$ and ${}_{NL}\boldsymbol{\beta}$.

Figure 2: 95% HPD of α and β under the linear and nonlinear LC model for the United Kingdom dataset.

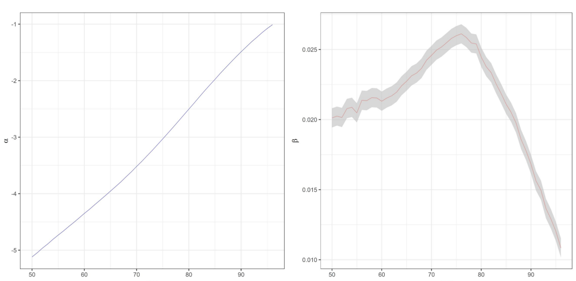


Linear estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

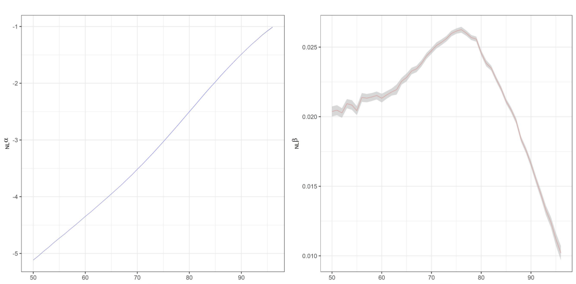


Nonlinear estimation of ${}_{NL}\boldsymbol{\alpha}$ and ${}_{NL}\boldsymbol{\beta}$.

Figure 3: 95% HPD of α and β under the linear and nonlinear LC model for the Italy dataset.

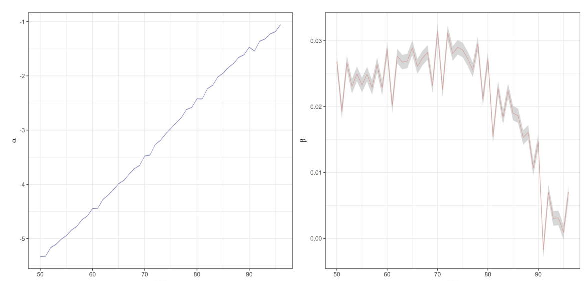


Linear estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

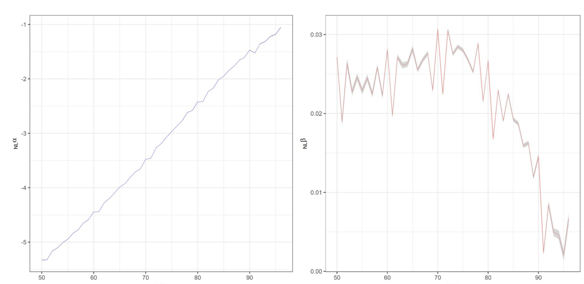


Nonlinear estimation of ${}_{NL}\boldsymbol{\alpha}$ and ${}_{NL}\boldsymbol{\beta}$.

Figure 4: 95% HPD of α and β under the linear and nonlinear LC model for the France dataset.

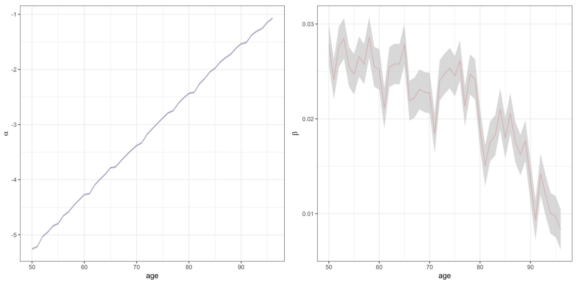


Linear estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

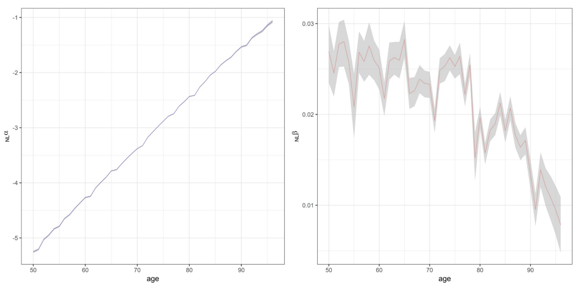


Nonlinear estimation of ${}_{NL}\boldsymbol{\alpha}$ and ${}_{NL}\boldsymbol{\beta}$.

Figure 5: 95% HPD of α and β under the linear and nonlinear LC model for the Spain dataset.

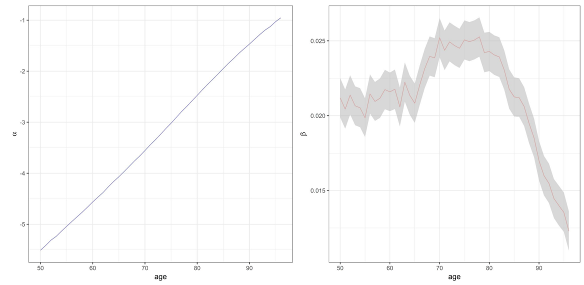


Linear estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

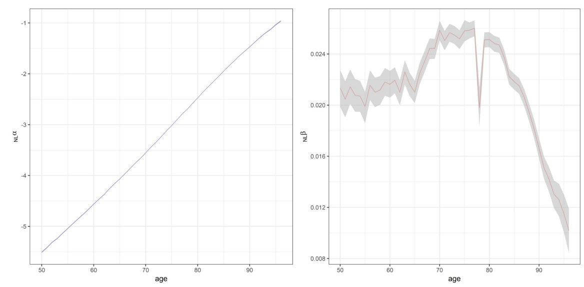


Nonlinear estimation of ${}_{NL}\boldsymbol{\alpha}$ and ${}_{NL}\boldsymbol{\beta}$.

Figure 6: 95% HPD of α and β under the linear and nonlinear LC model for the New-Zealand dataset.

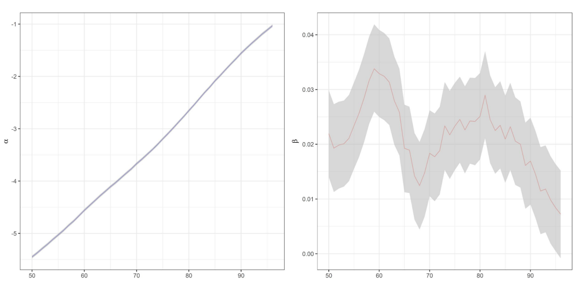


Linear estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

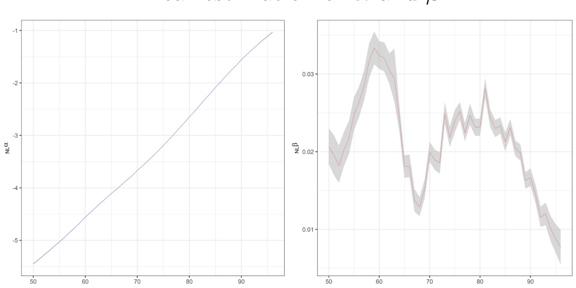


Nonlinear estimation of ${}_{NL}\boldsymbol{\alpha}$ and ${}_{NL}\boldsymbol{\beta}$.

Figure 7: 95% HPD of α and β under the linear and nonlinear LC model for the Sweden dataset.

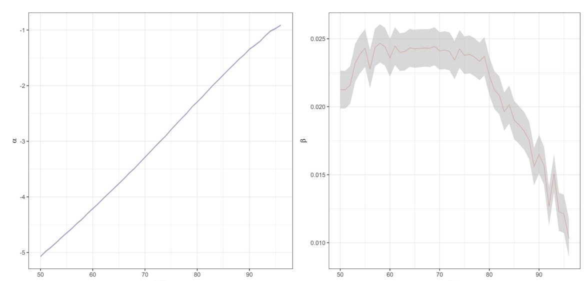


Linear estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

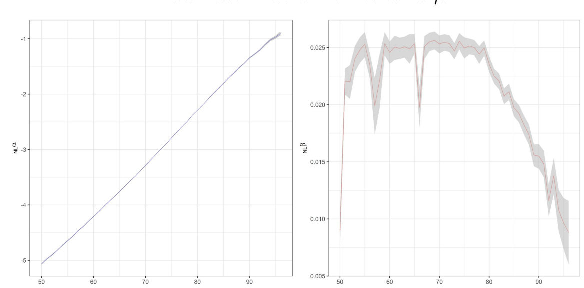


Nonlinear estimation of ${}_{NL}\boldsymbol{\alpha}$ and ${}_{NL}\boldsymbol{\beta}$.

Figure 8: 95% HPD of α and β under the linear and nonlinear LC model for the Germany dataset



Linear estimation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.



Nonlinear estimation of ${}_{NL}\boldsymbol{\alpha}$ and ${}_{NL}\boldsymbol{\beta}$.

Figure 9: 95% HPD of α and β under the linear and nonlinear LC model for the Finland dataset

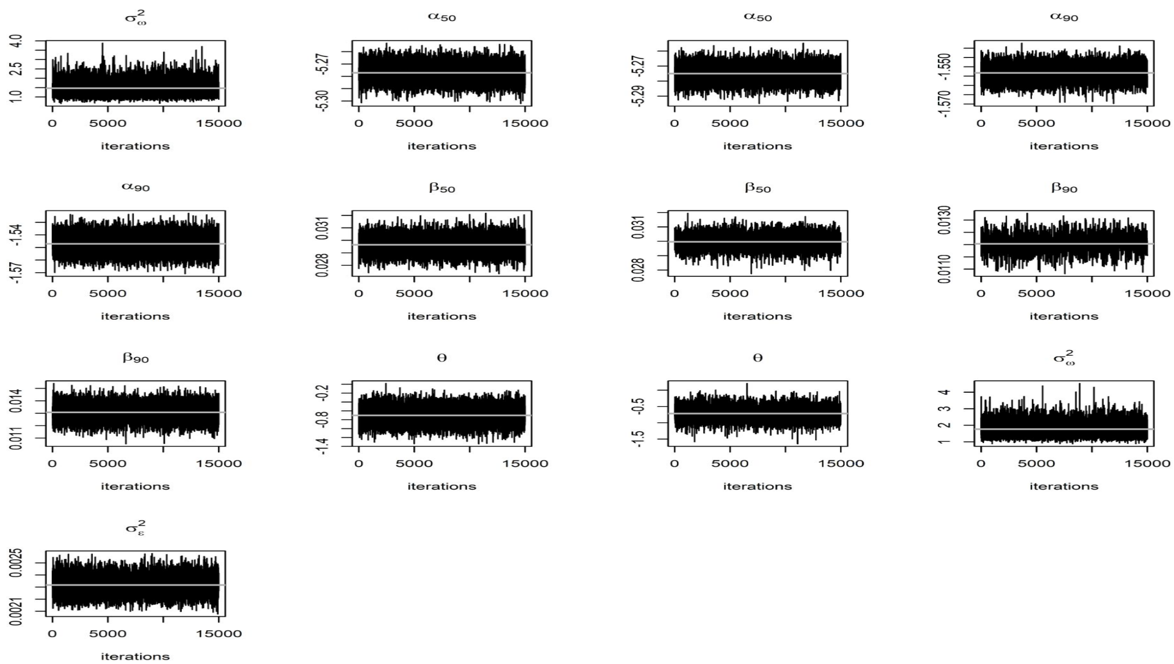


Figure 10: Static parameter trace plot for the linear and nonlinear LC model for the Australian dataset.

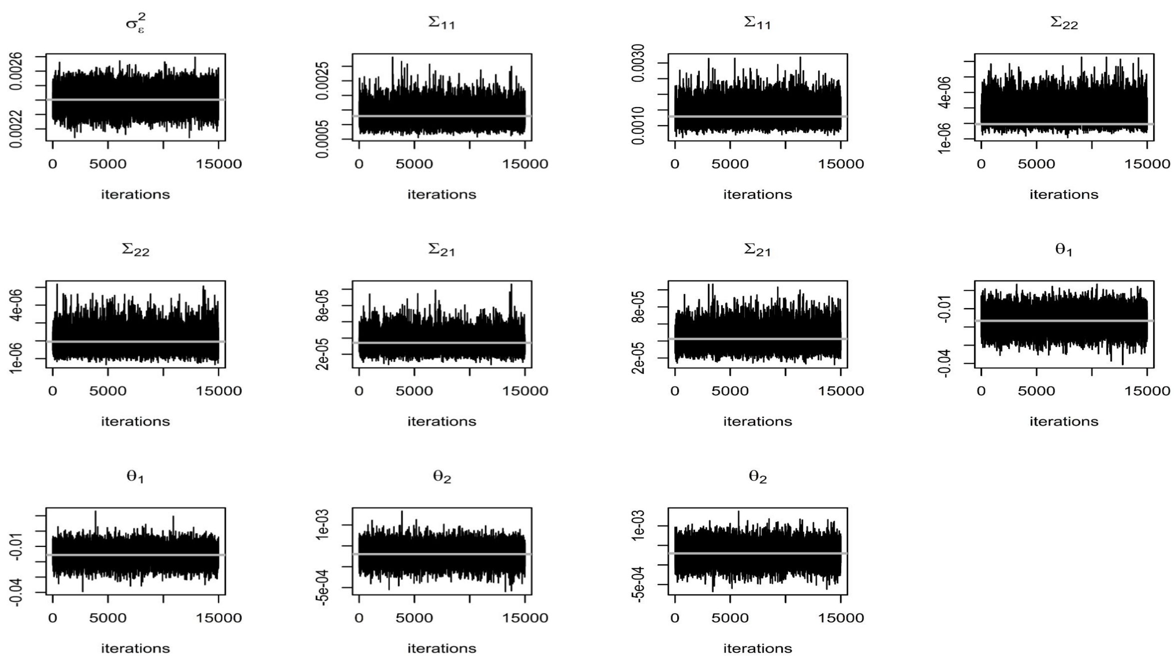


Figure 11: Static parameter trace plot for the linear and nonlinear CBD model for the Australian dataset.

1.7. Bayesian model: latent factor analysis and forecast.

Figure 12: Posterior distribution of κ_t under the LC model for the Australian dataset.

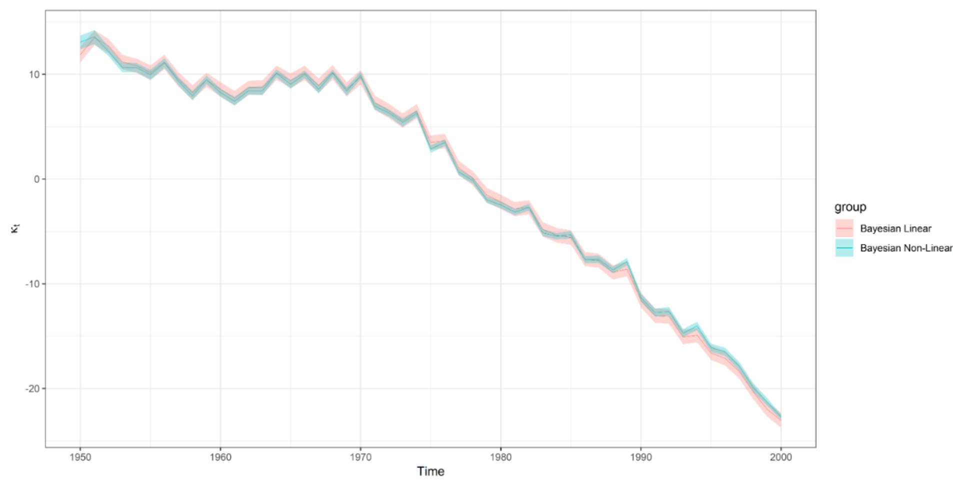
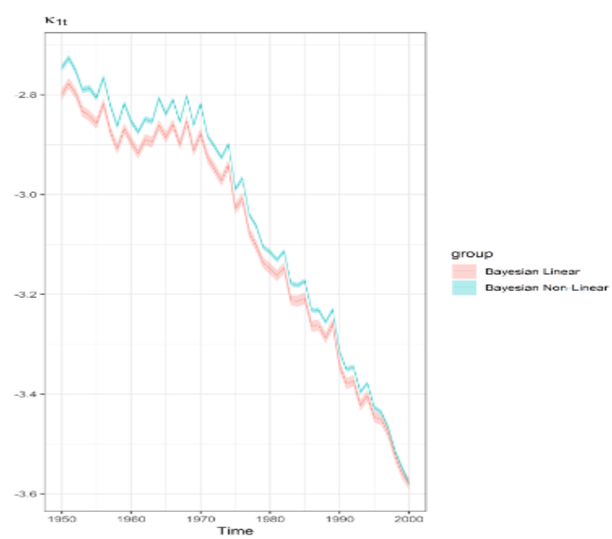


Figure 13: Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model for the Australian dataset.



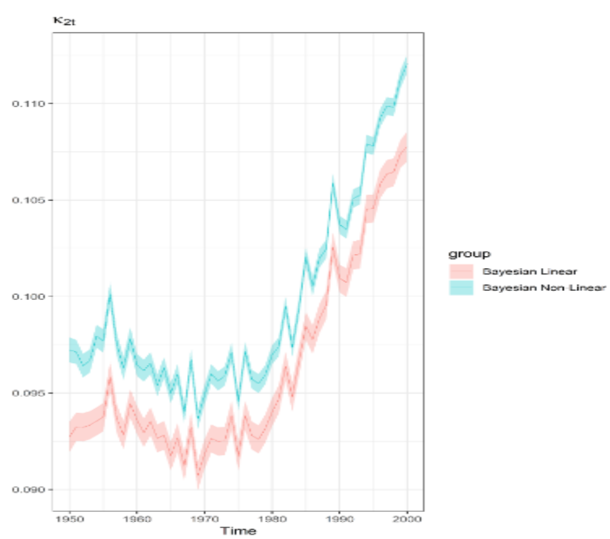


Figure 14: 10-year forecast, κ_t under LC with a 95% credible interval range for the Australian dataset.

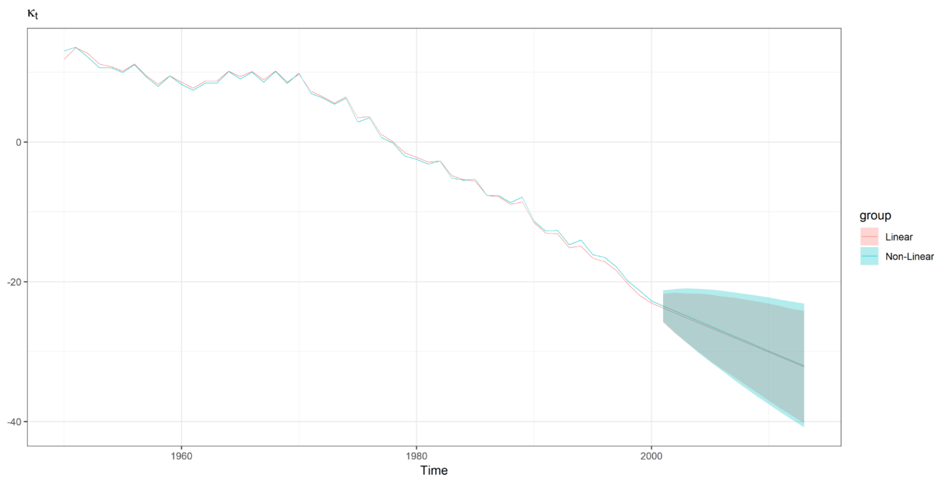
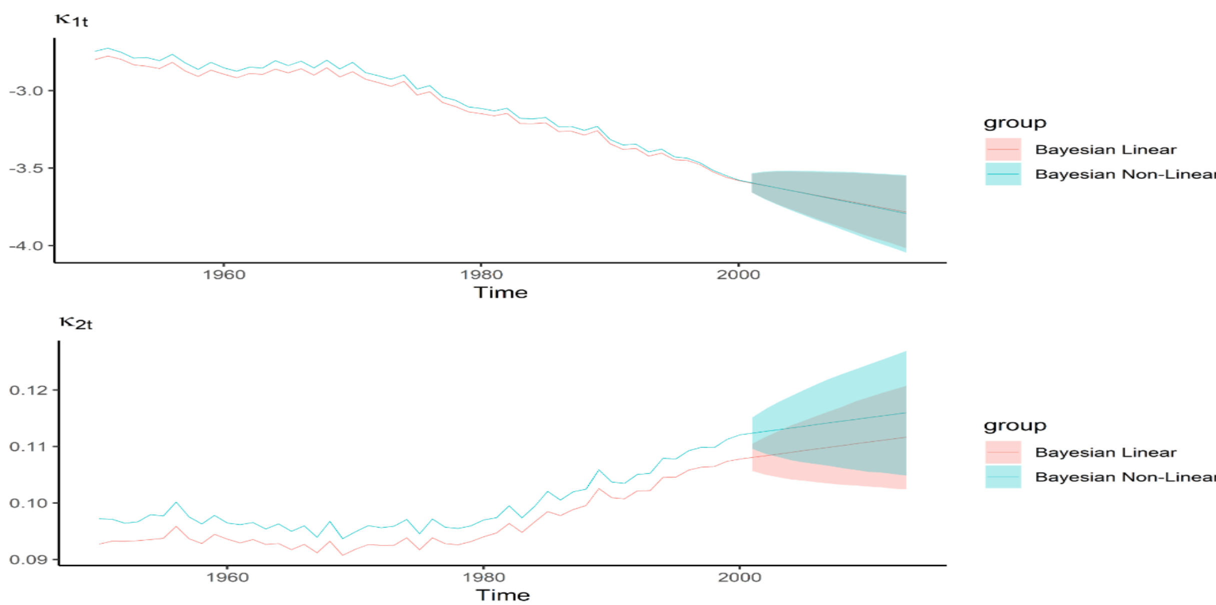
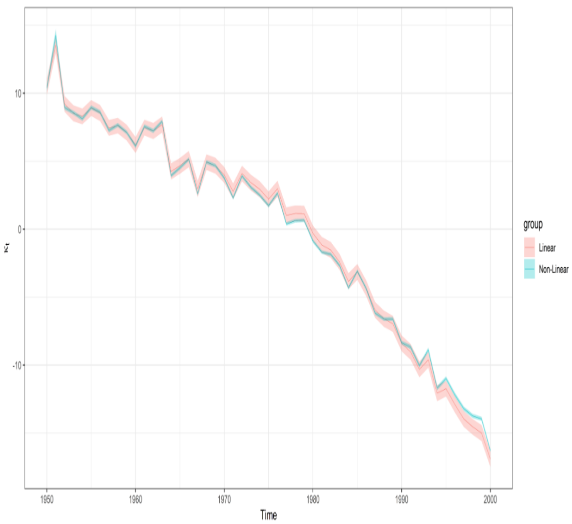
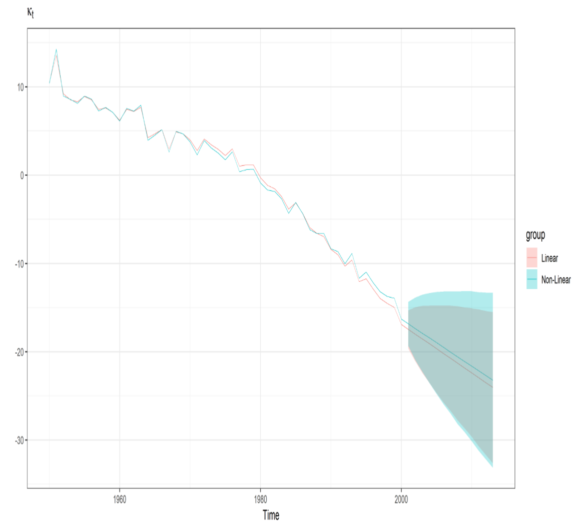


Figure 15: 10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range for the Australian dataset.



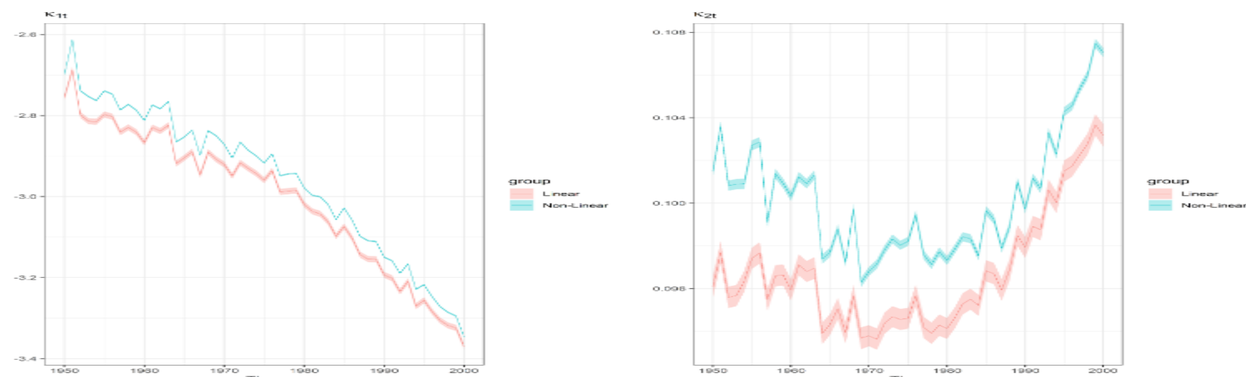


Posterior distribution of κ_t under the LC model.

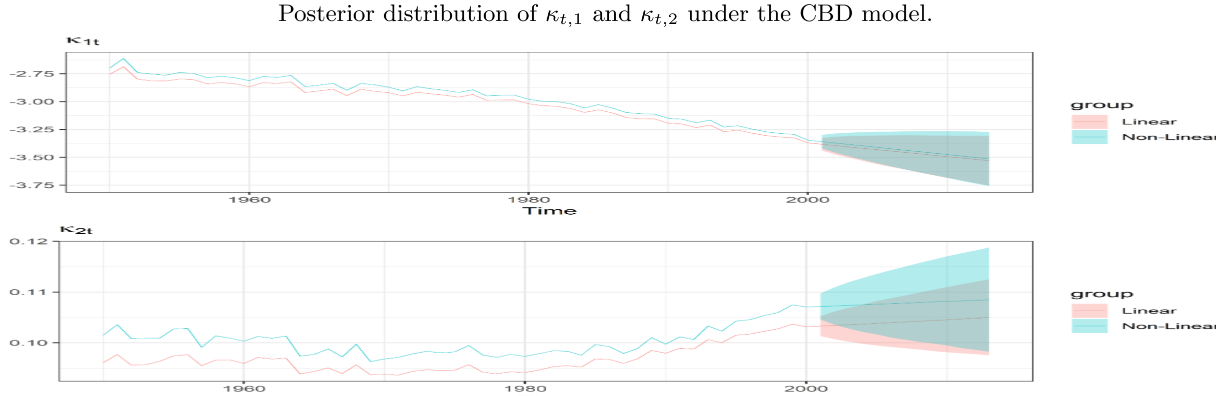


10-year forecast, κ_t under LC model with a 95% credible interval range.

Figure 16: Posterior distribution and 10 year forecasts for κ_t under the LC model using the United Kingdom mortality dataset.

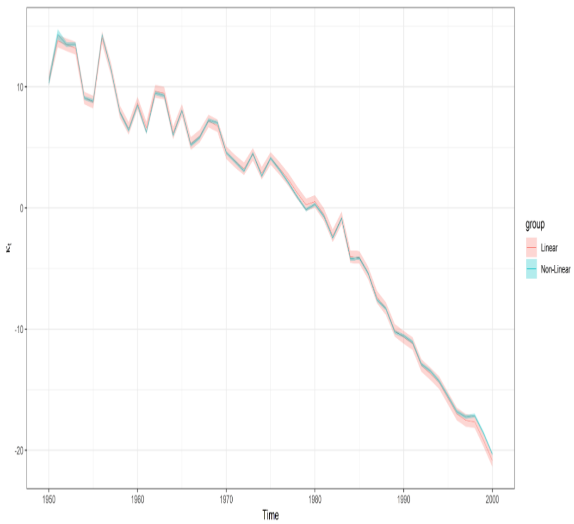


Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model.

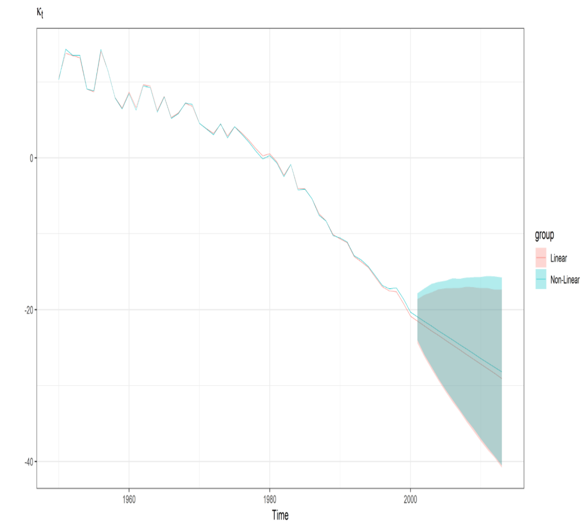


10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range.

Figure 17: Posterior distribution and 10 year forecasts for $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model using the United Kingdom mortality dataset.

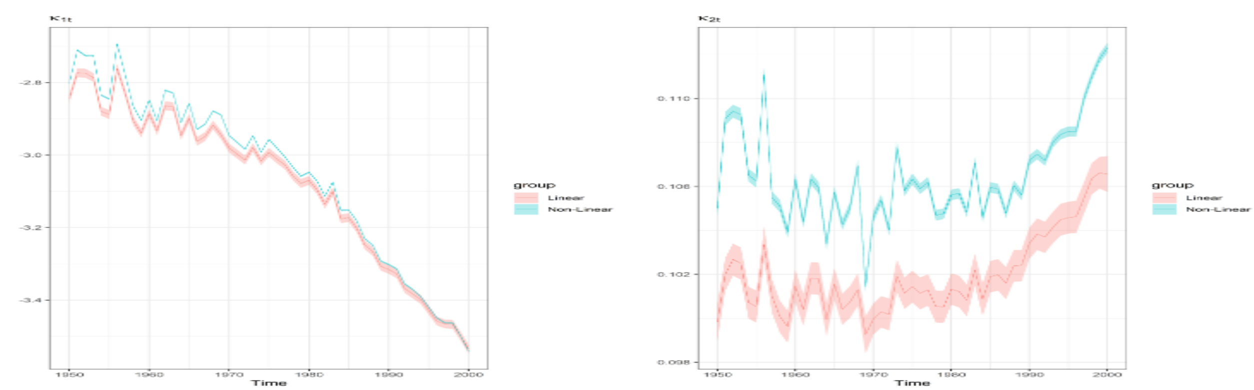


Posterior distribution of κ_t under the LC model.

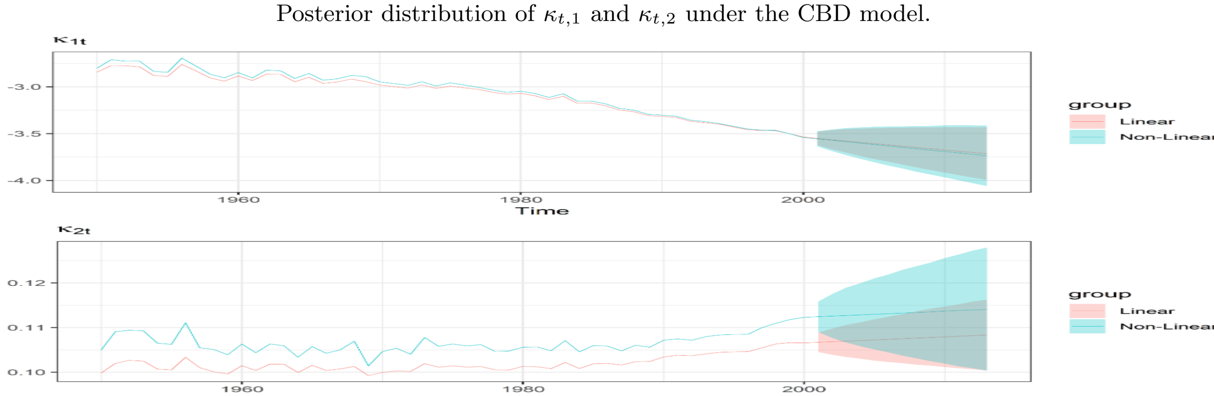


10-year forecast, κ_t under LC with a 95% credible interval range.

Figure 18: Posterior distribution and 10 year forecasts for κ_t under the LC model using the Italy mortality dataset.

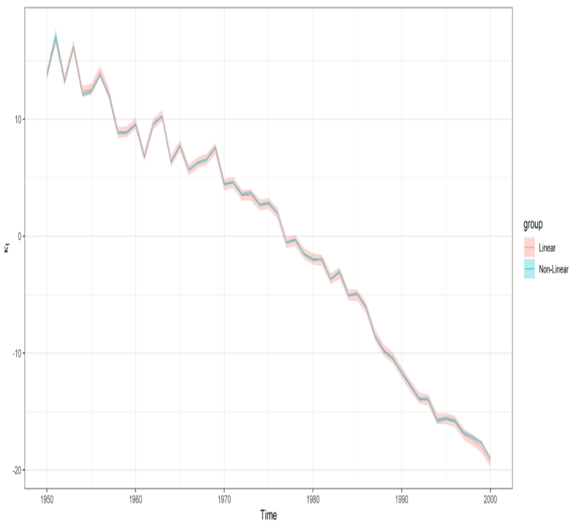


Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model.

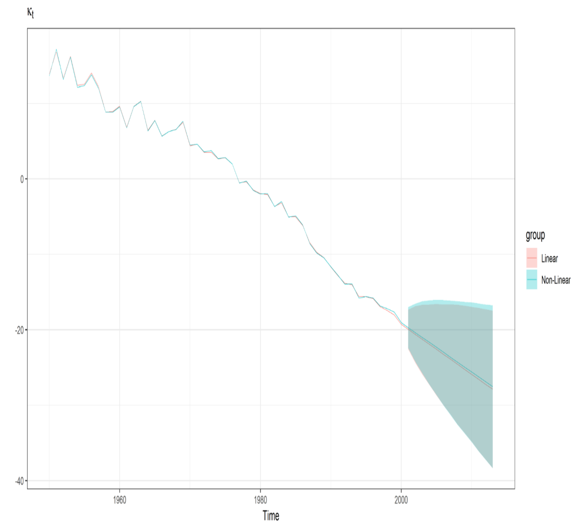


10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range.

Figure 19: Posterior distribution and 10 year forecasts for $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model using the Italy mortality dataset.

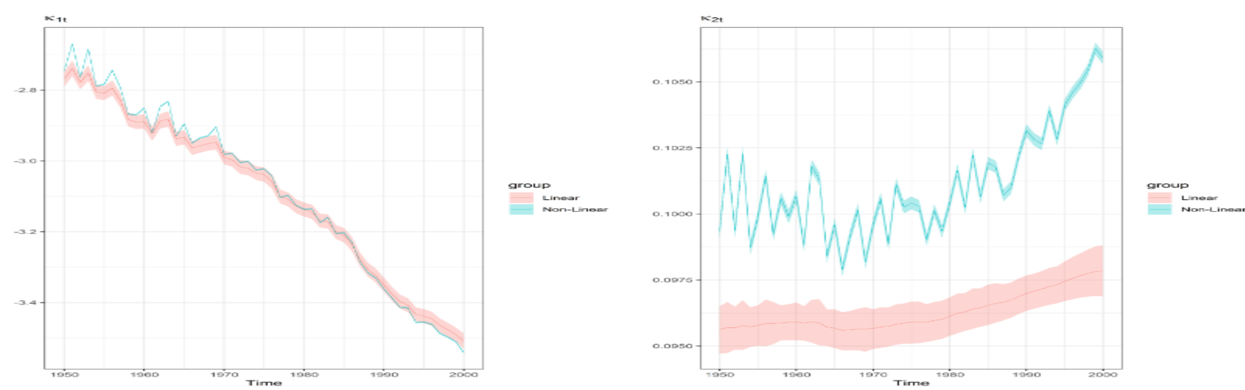


Posterior distribution of κ_t under the LC model.

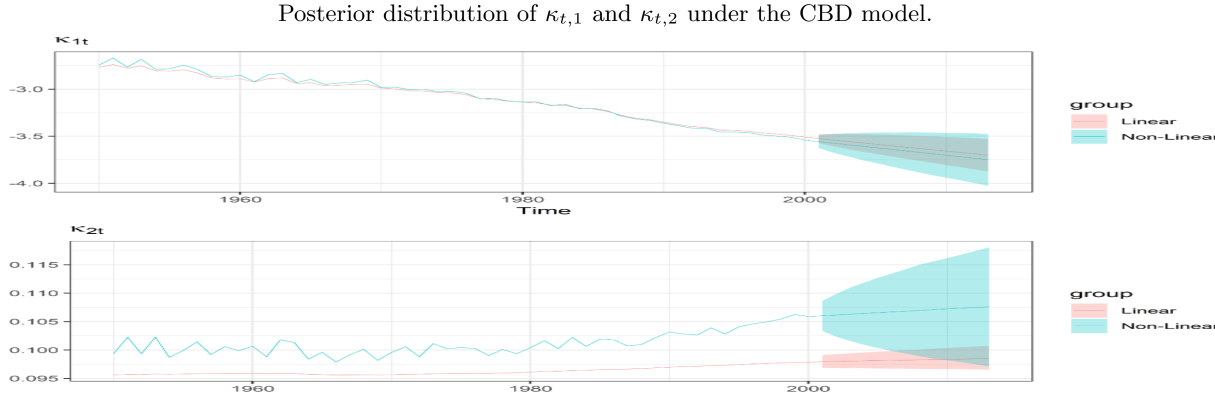


10-year forecast, κ_t under LC with a 95% credible interval range.

Figure 20: Posterior distribution and 10 year forecasts for κ_t under the LC model using the France mortality dataset.

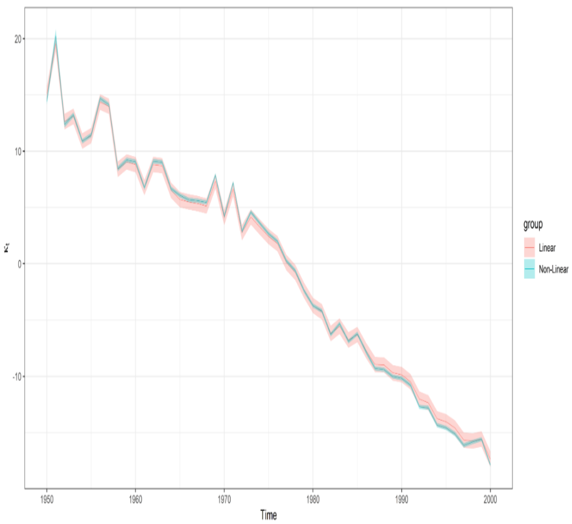


Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model.

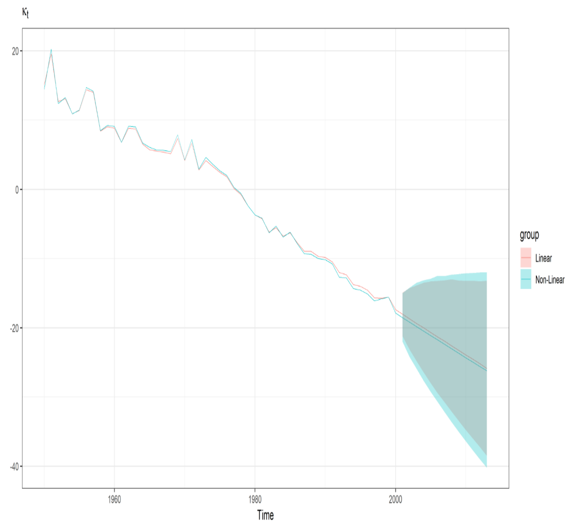


10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range.

Figure 21: Posterior distribution and 10 year forecasts for $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model using the France mortality dataset.

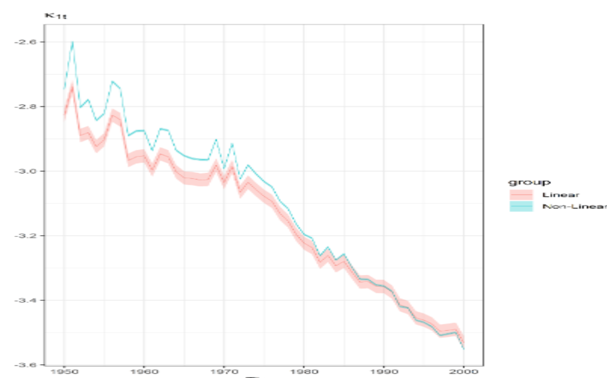


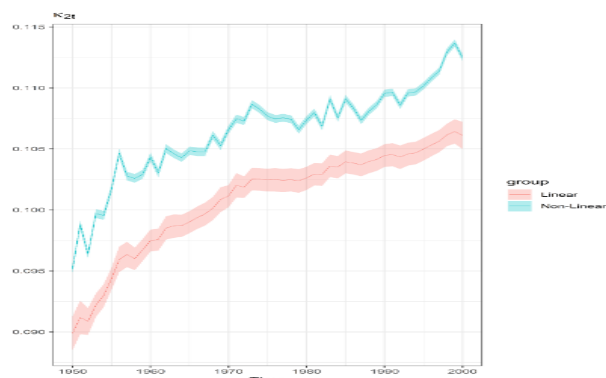
Posterior distribution of κ_t under the LC model.



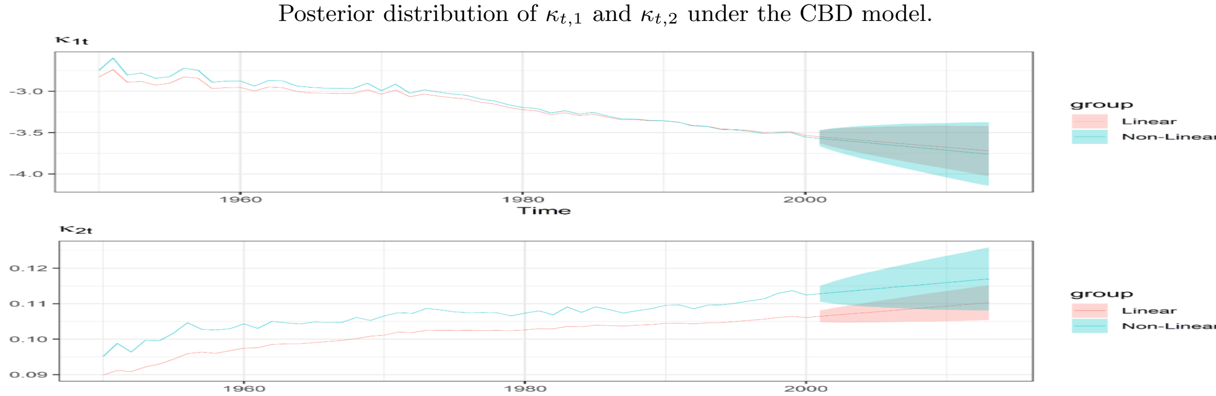
10-year forecast, κ_t under LC with a 95% credible interval range.

Figure 22: Posterior distribution and 10 year forecasts for κ_t under the LC model using the Spain mortality dataset.



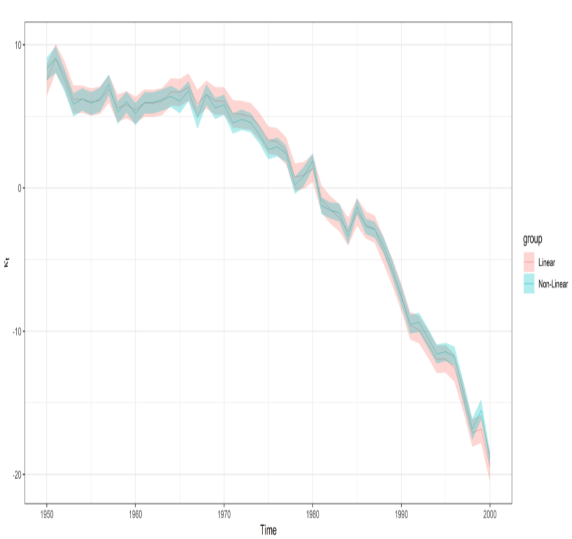


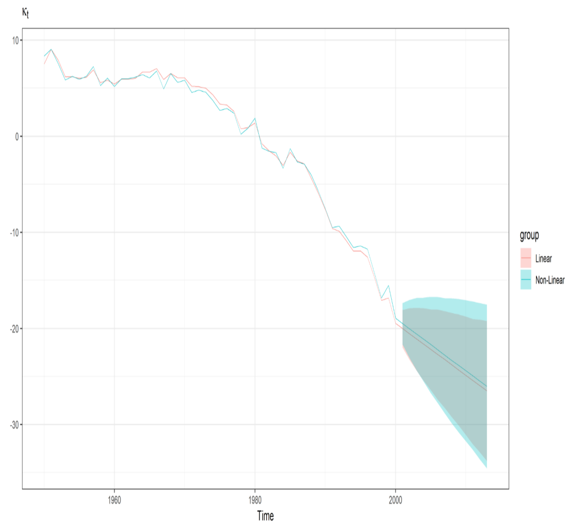
Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model.



10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range.

Figure 23: Posterior distribution and 10 year forecasts for $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model using the Spain mortality dataset.

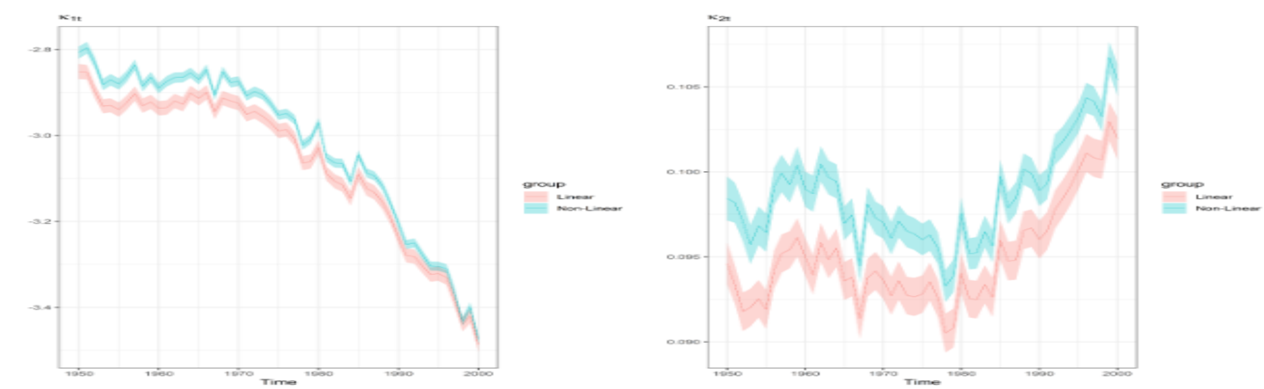




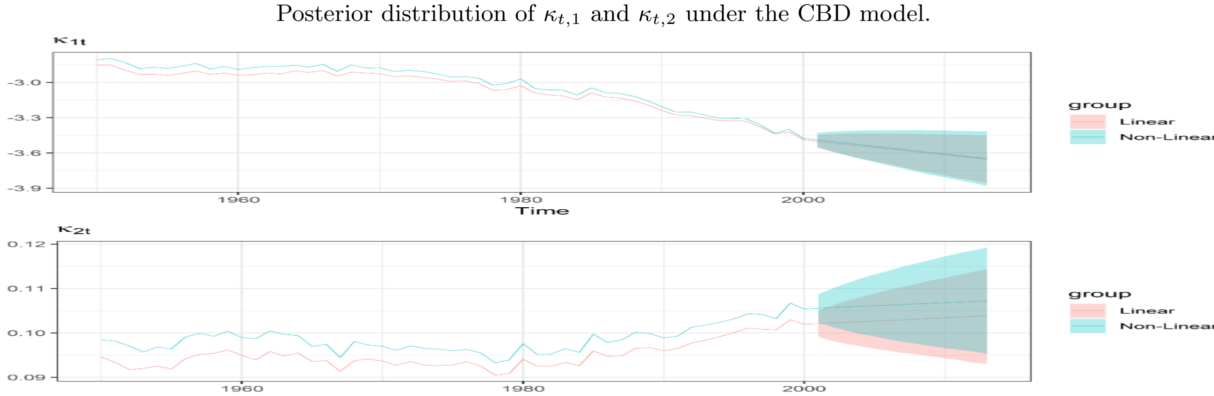
Posterior distribution of κ_t under the LC model.

10-year forecast, κ_t under LC with a 95% credible interval range.

Figure 24: Posterior distribution and 10 year forecasts for κ_t under the LC model using the New Zealand mortality dataset.

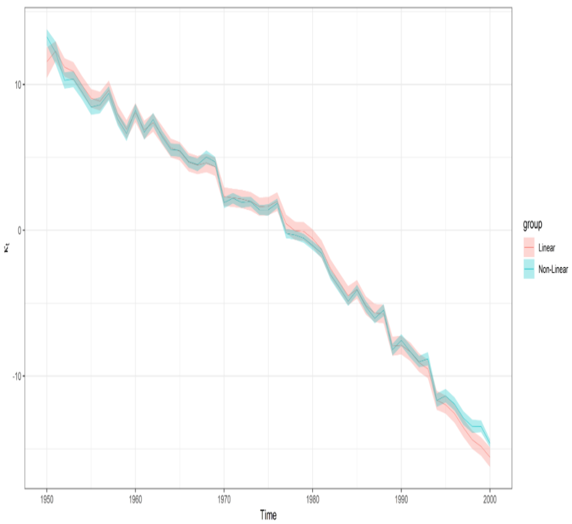


Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model.

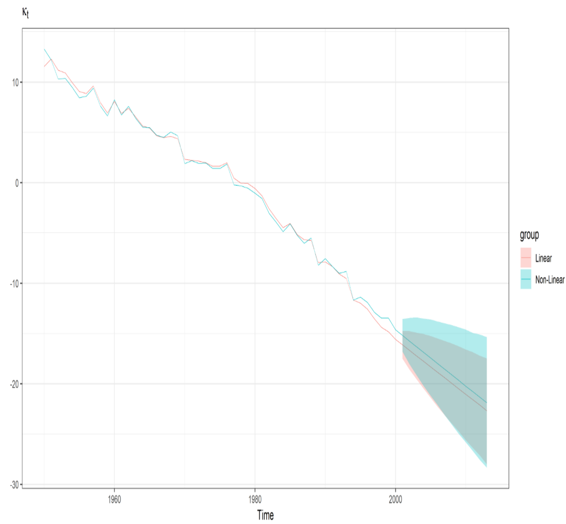


10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range.

Figure 25: Posterior distribution and 10 year forecasts for $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model using the New Zealand mortality dataset.

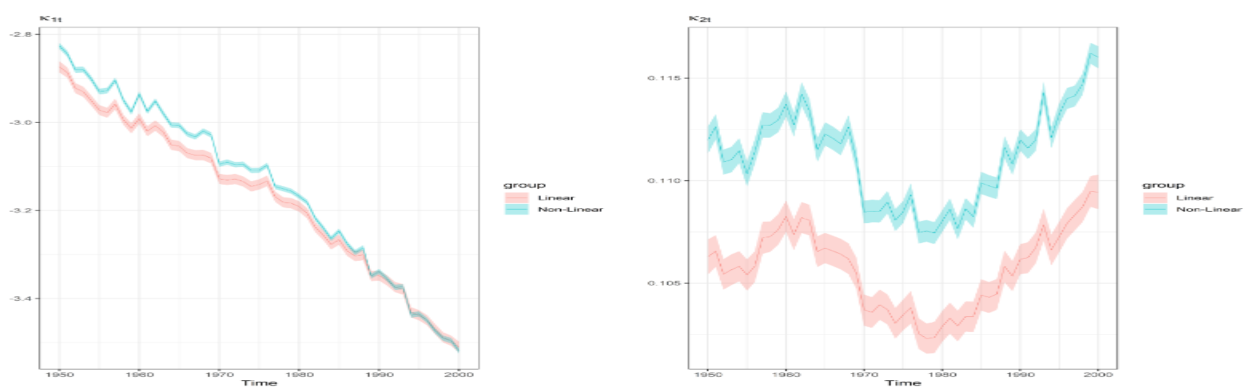


Posterior distribution of κ_t under the LC model.

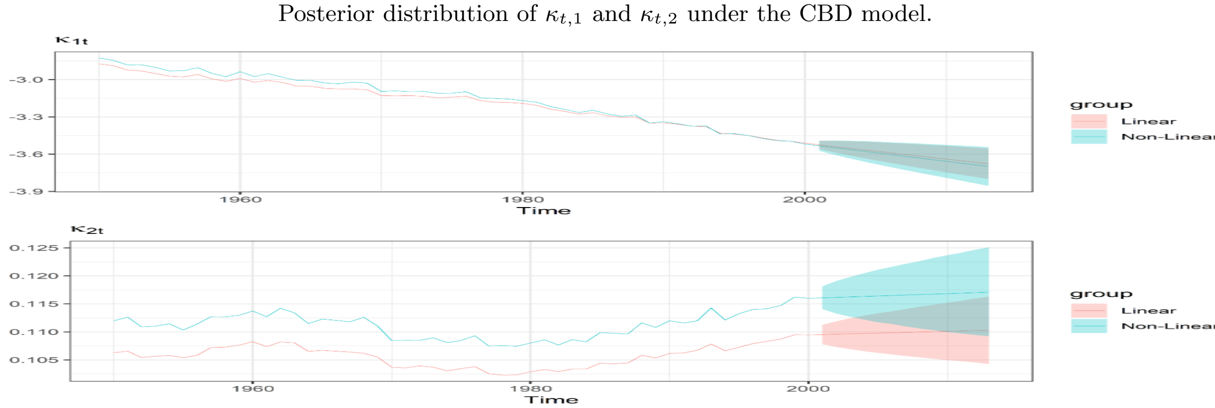


10-year forecast, κ_t under LC with a 95% credible interval range.

Figure 26: Posterior distribution and 10 year forecasts for κ_t under the LC model using the Sweden mortality dataset.

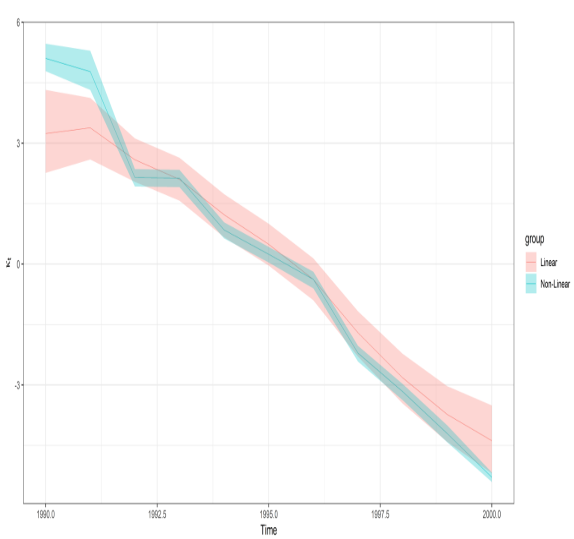


Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model.

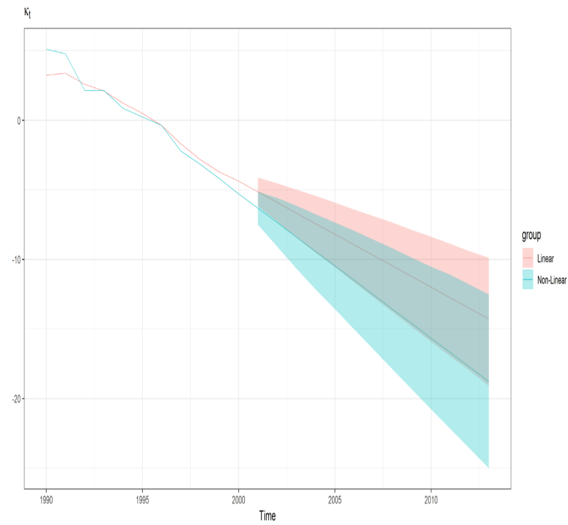


10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range.

Figure 27: Posterior distribution and 10 year forecasts for $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model using the Sweden mortality dataset.

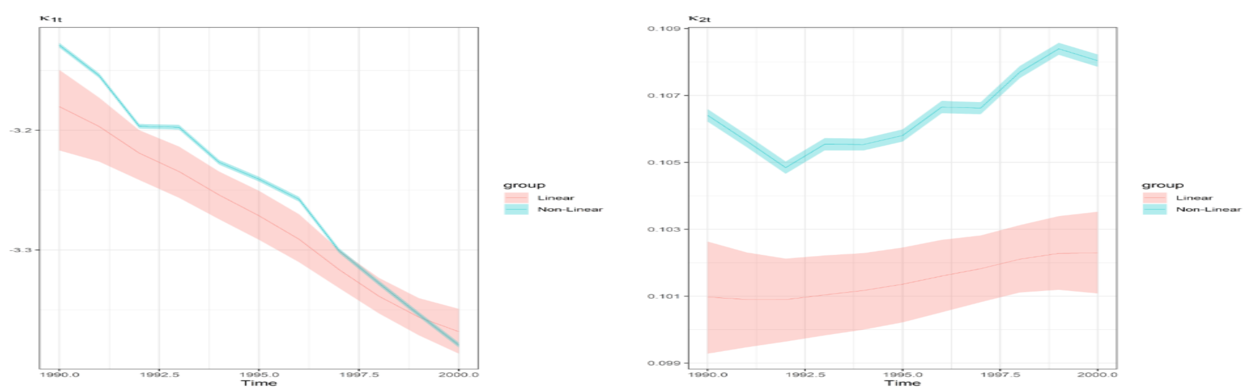


Posterior distribution of κ_t under the LC model.

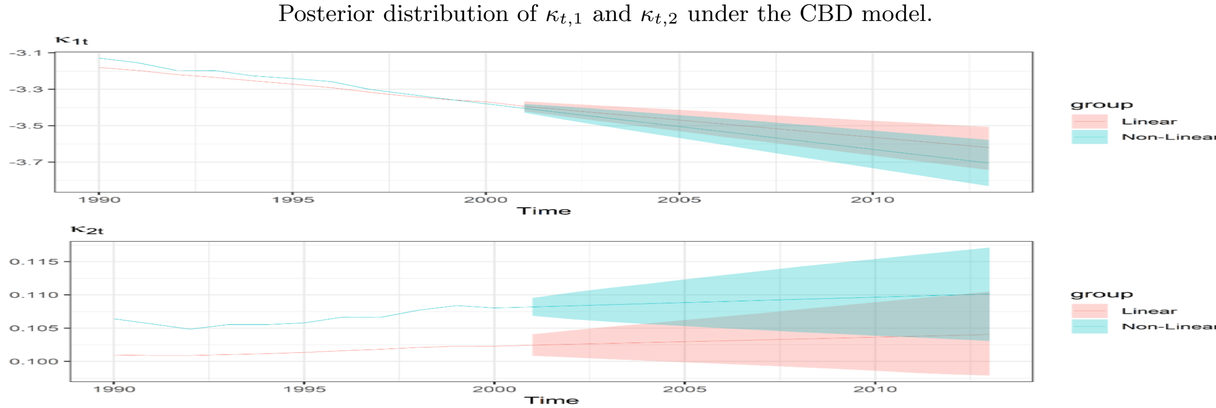


10-year forecast, κ_t under LC with a 95% credible interval range.

Figure 28: Posterior distribution and 10 year forecasts for κ_t under the LC model using the Germany mortality dataset.

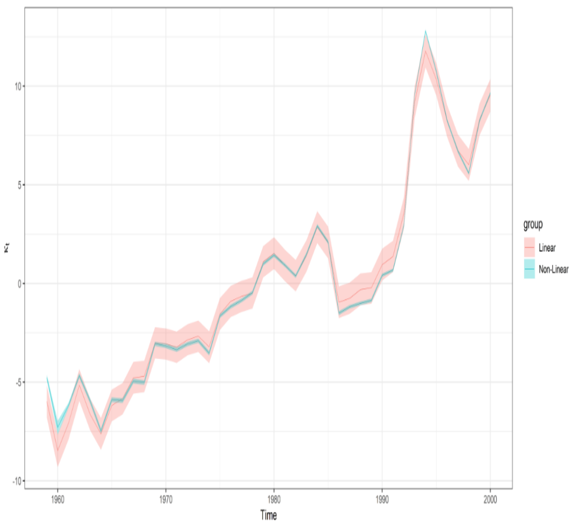


Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model.

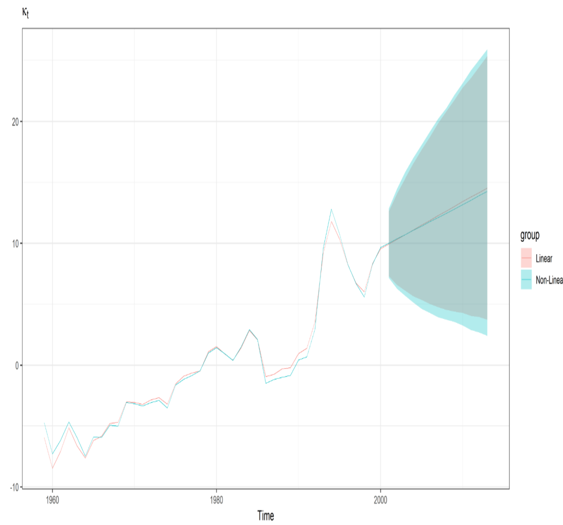


10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range.

Figure 29: Posterior distribution and 10 year forecasts for $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model using the Germany mortality dataset.

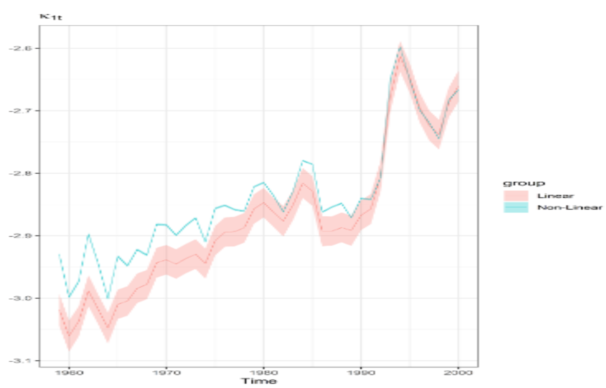


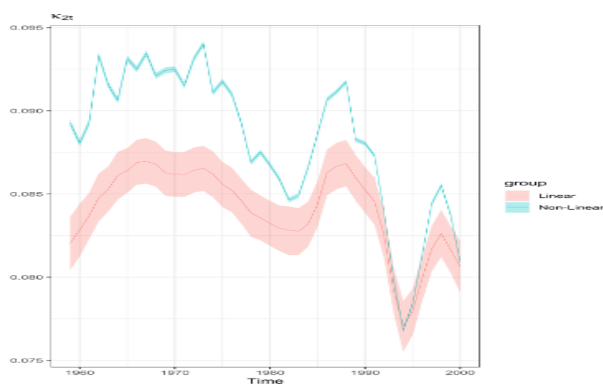
Posterior distribution of κ_t under the LC model.



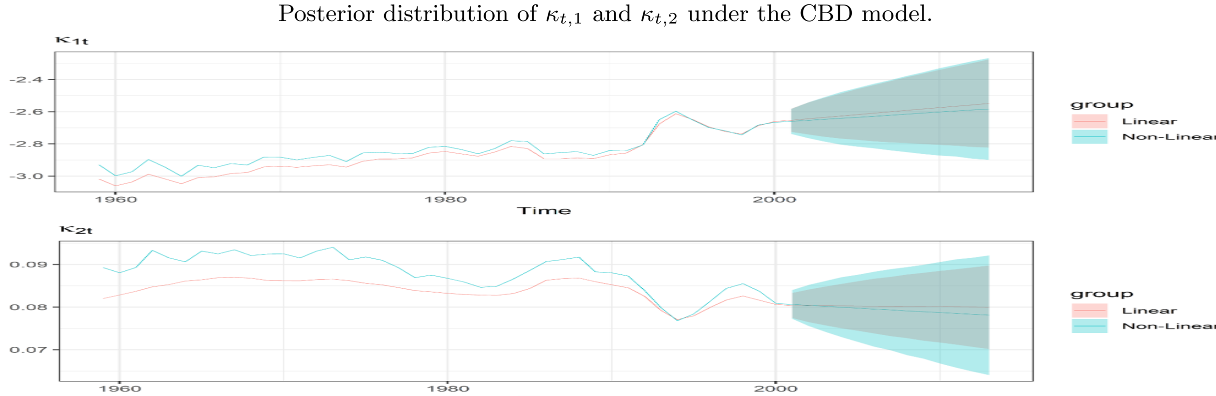
10-year forecast, κ_t under LC with a 95% credible interval range.

Figure 30: Posterior distribution and 10 year forecasts for κ_t under the LC model using the Russia mortality dataset.





Posterior distribution of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model.

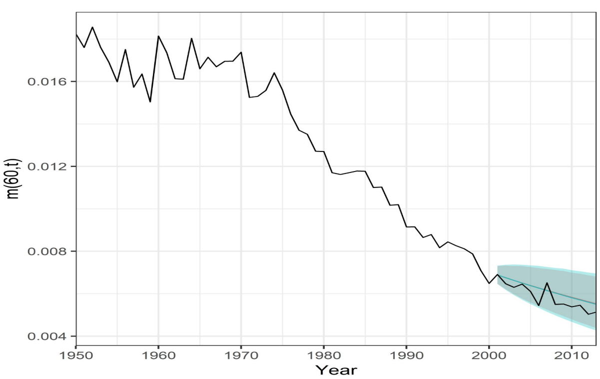


10-year forecast of $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model with a 95% credible interval range.

Figure 31: Posterior distribution and 10 year forecasts for $\kappa_{t,1}$ and $\kappa_{t,2}$ under the CBD model using the Russia mortality dataset.

(a) $m(50,t)$ Forecasts

(d) $m(80,t)$ Forecasts



(b) $m(60,t)$ Forecasts

(e) $m(90,t)$ Forecasts

(c) $m(70,t)$ Forecasts

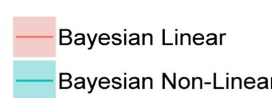
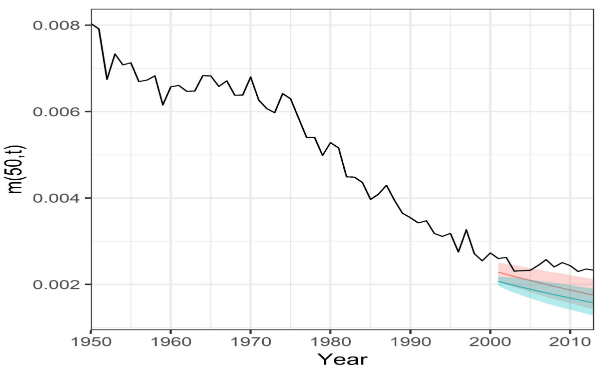
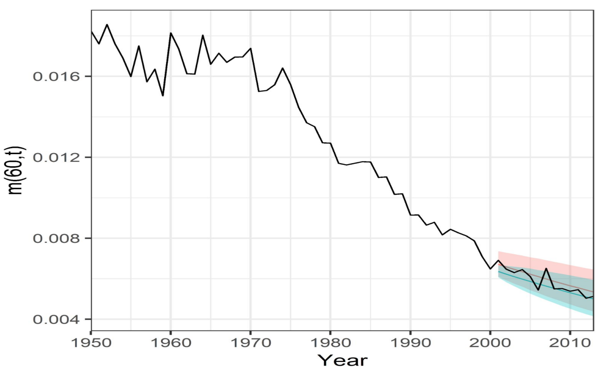


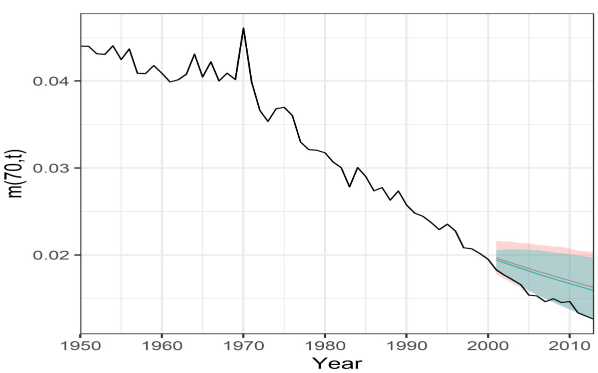
Figure 32: Forecasts over period 2001-2013 for the Australian dataset under the LC model and its variants.



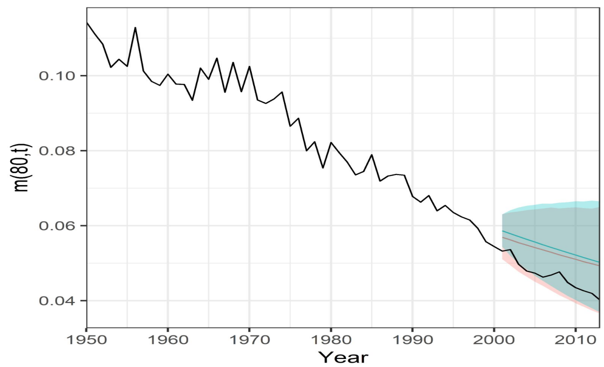
(a) $m(50, t)$ Forecasts



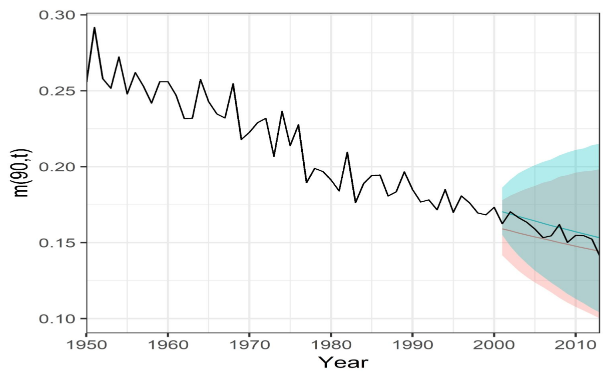
(b) $m(60, t)$ Forecasts



(c) $m(70, t)$ Forecasts



(d) $m(80, t)$ Forecasts



(e) $m(90, t)$ Forecasts

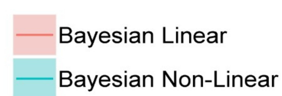
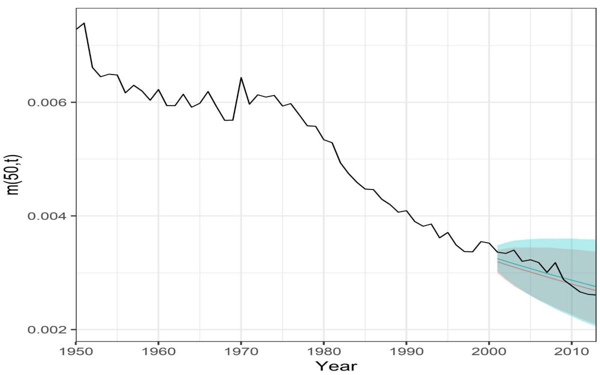
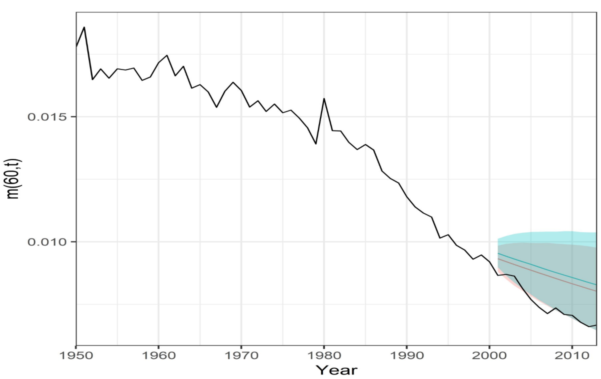


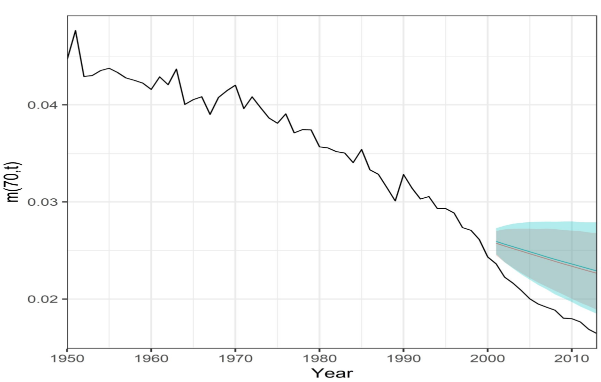
Figure 33: Forecasts over period 2001-2013 for the Australian dataset under the CBD model and its variants.



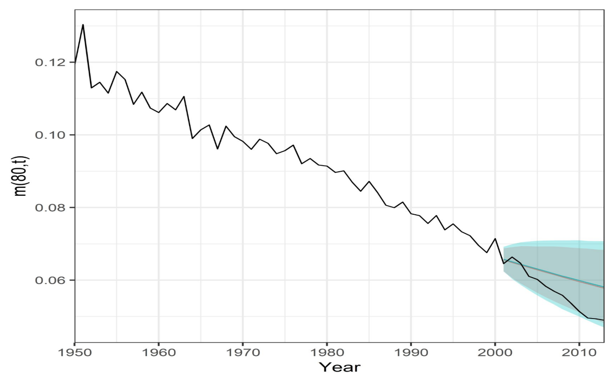
(a) $m(50,t)$ Forecasts



(b) $m(60,t)$ Forecasts



(c) $m(70,t)$ Forecasts



(d) $m(80,t)$ Forecasts

(e) $m(90,t)$ Forecasts

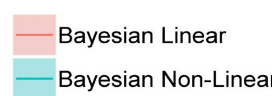
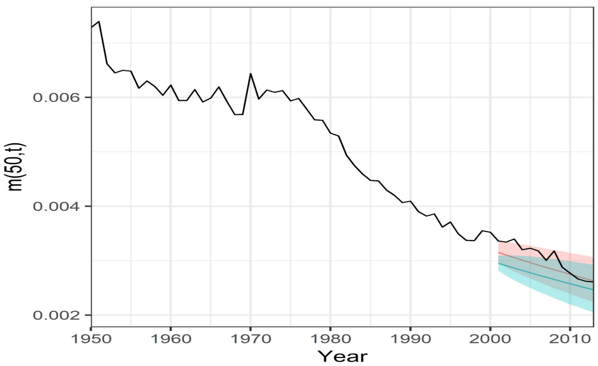
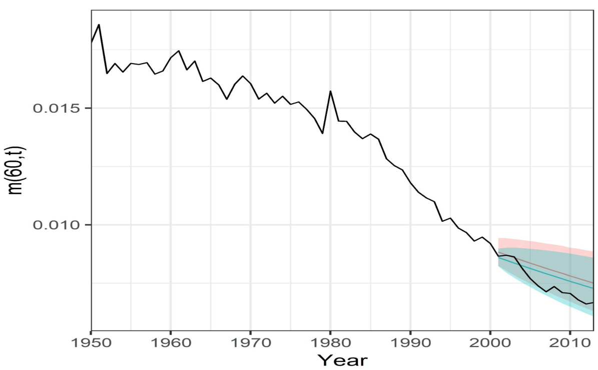


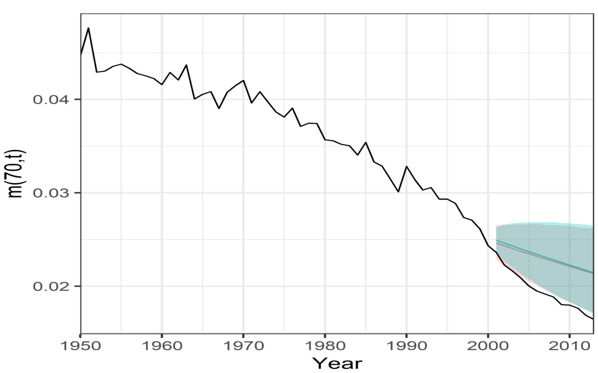
Figure 34: Forecasts over period 2001-2013 for the United Kingdom dataset under the LC model and its variants.



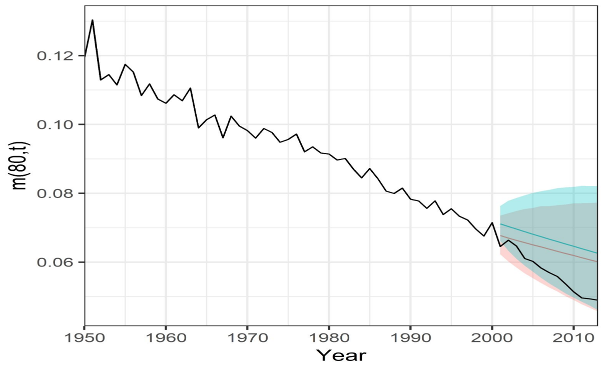
(a) $m(50,t)$ Forecasts



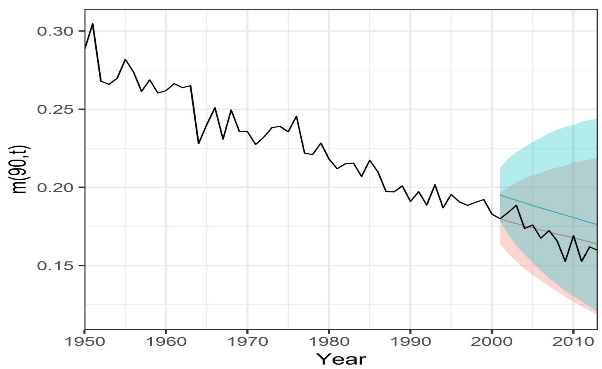
(b) $m(60,t)$ Forecasts



(c) $m(70,t)$ Forecasts



(d) $m(80,t)$ Forecasts



(e) $m(90,t)$ Forecasts

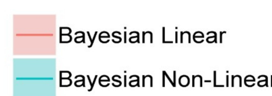
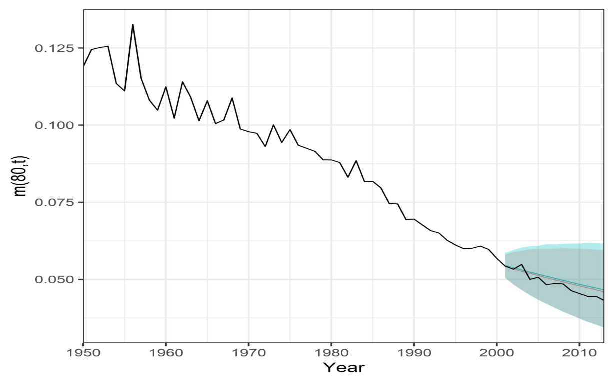
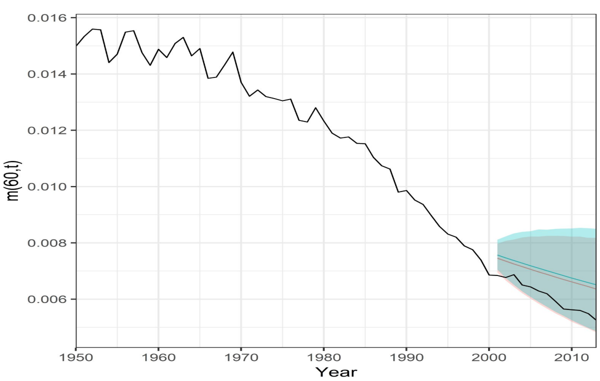


Figure 35: Forecasts over period 2001-2013 for the United Kingdom dataset under the CBD model and its variants.

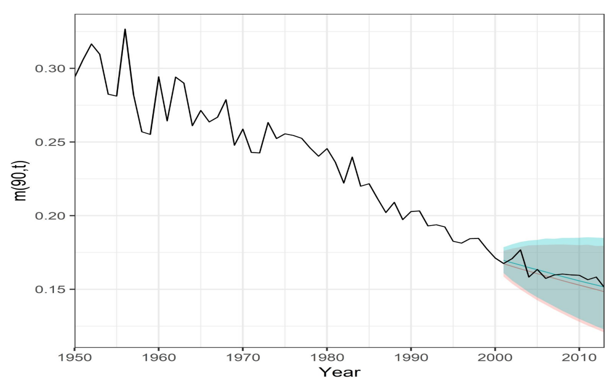
(a) $m(50,t)$ Forecasts



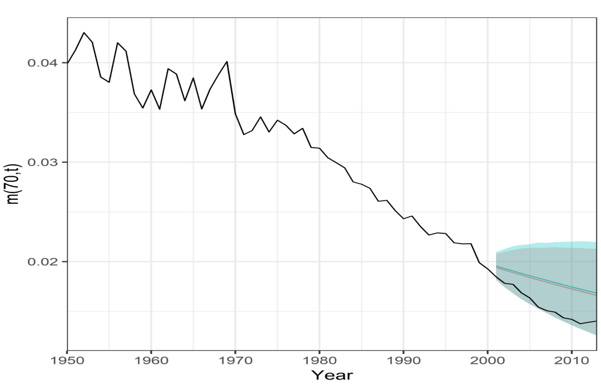
(d) $m(80,t)$ Forecasts



(b) $m(60,t)$ Forecasts



(e) $m(90,t)$ Forecasts



(c) $m(70,t)$ Forecasts

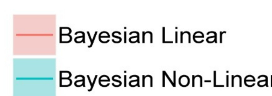
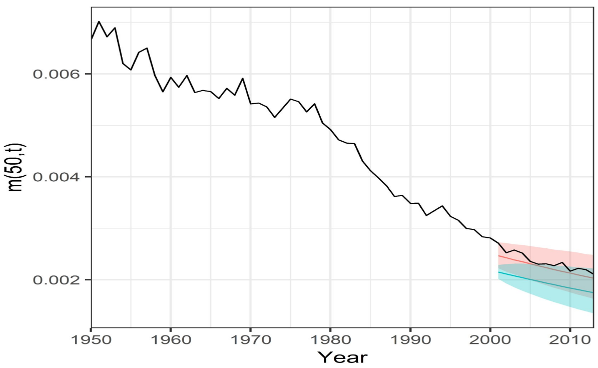
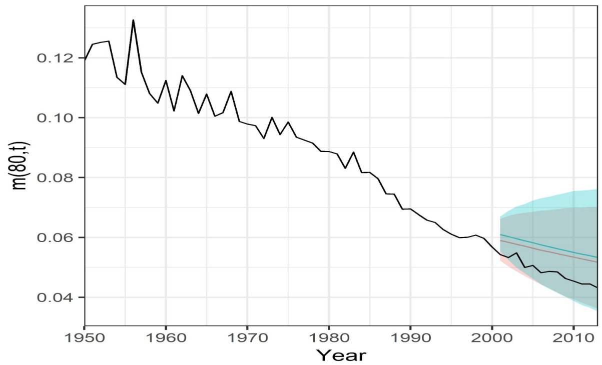


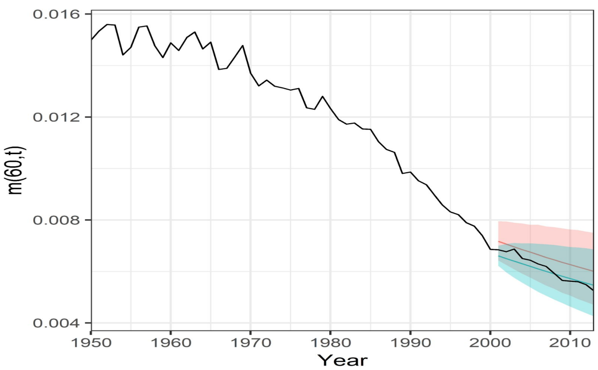
Figure 36: Forecasts over period 2001-2013 for the Italy dataset under the LC model and its variants.



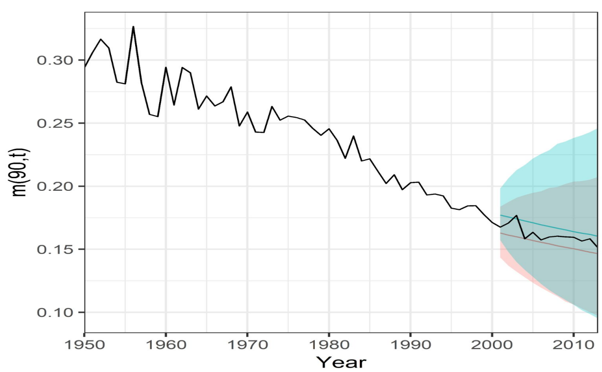
(a) $m(50,t)$ Forecasts



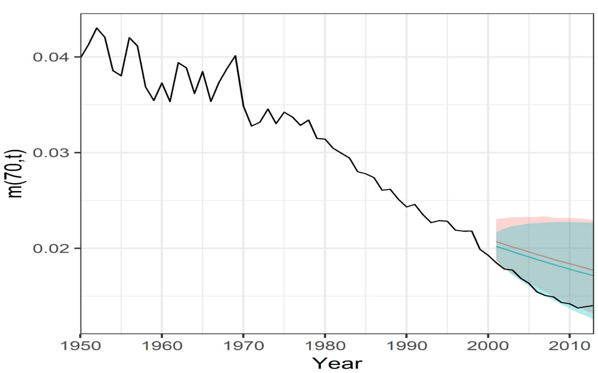
(d) $m(80,t)$ Forecasts



(b) $m(60,t)$ Forecasts



(e) $m(90,t)$ Forecasts



(c) $m(70,t)$ Forecasts

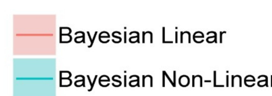
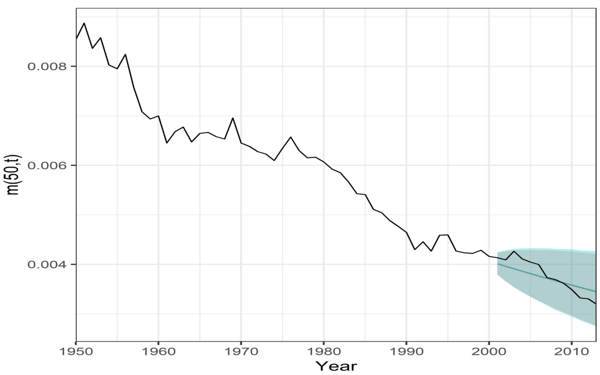
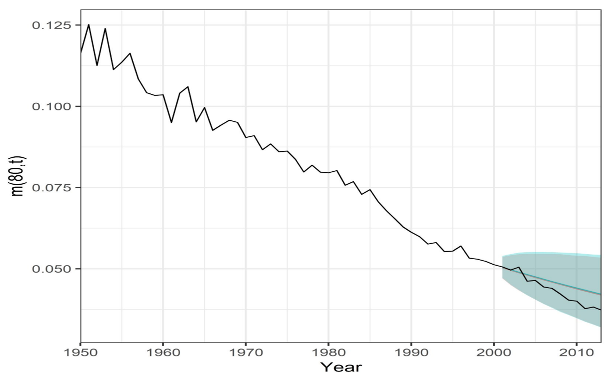


Figure 37: Forecasts over period 2001-2013 for the Italy dataset under the CBD model and its variants.



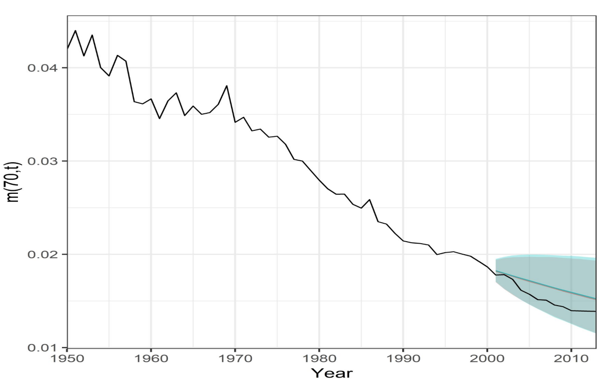
(a) $m(50, t)$ Forecasts



(d) $m(80, t)$ Forecasts

(b) $m(60, t)$ Forecasts

(e) $m(90, t)$ Forecasts



(c) $m(70, t)$ Forecasts

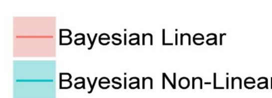
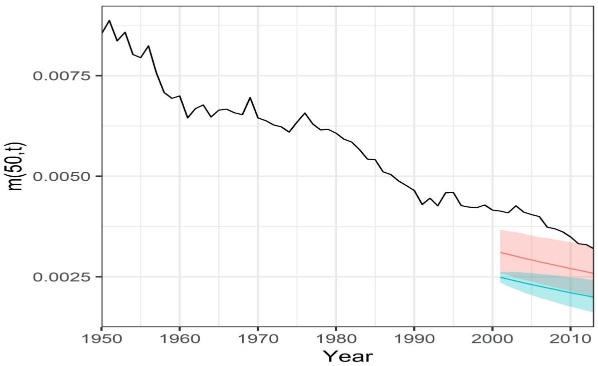
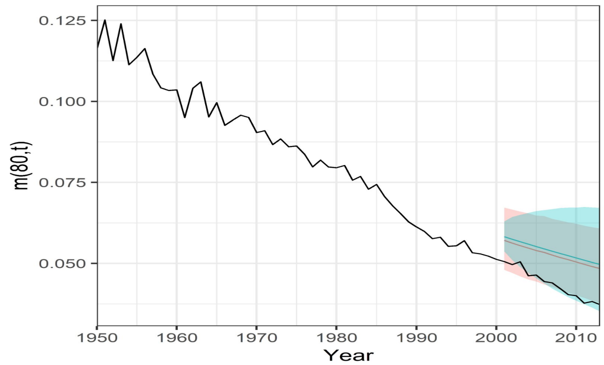


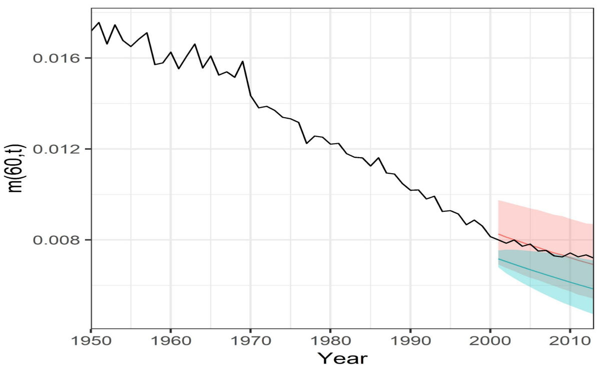
Figure 38: Forecasts over period 2001-2013 for the France dataset under the LC model and its variants.



(a) $m(50, t)$ Forecasts

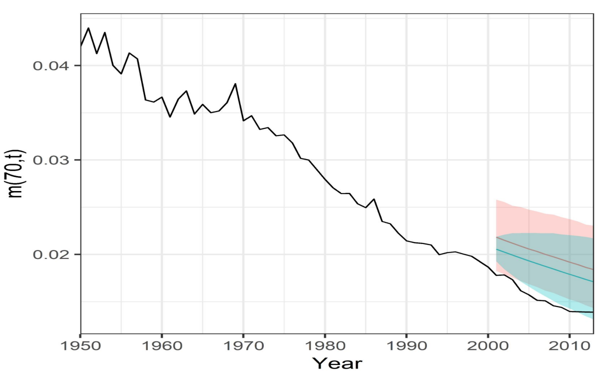


(d) $m(80, t)$ Forecasts



(b) $m(60, t)$ Forecasts

(e) $m(90, t)$ Forecasts



(c) $m(70, t)$ Forecasts

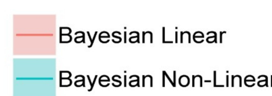
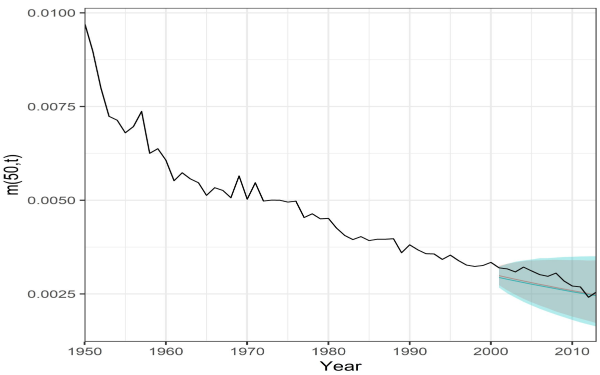
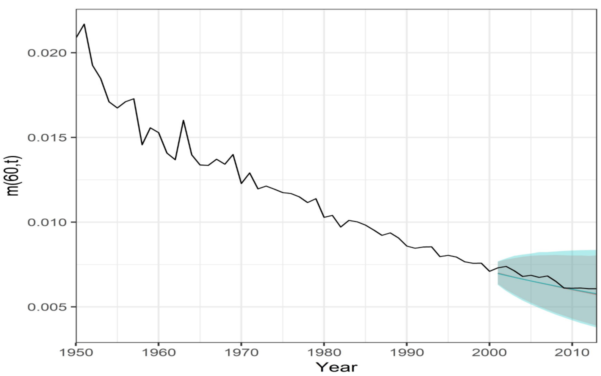


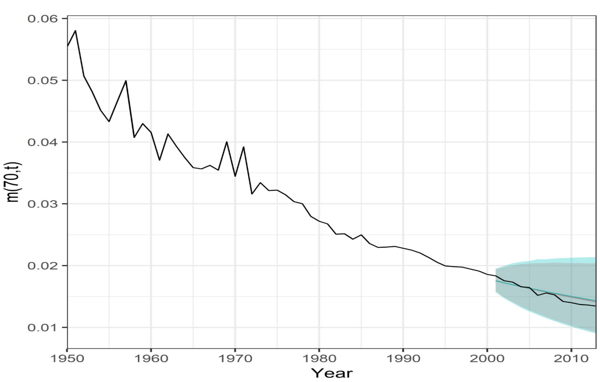
Figure 39: Forecasts over period 2001-2013 for the France dataset under the CBD model and its variants.



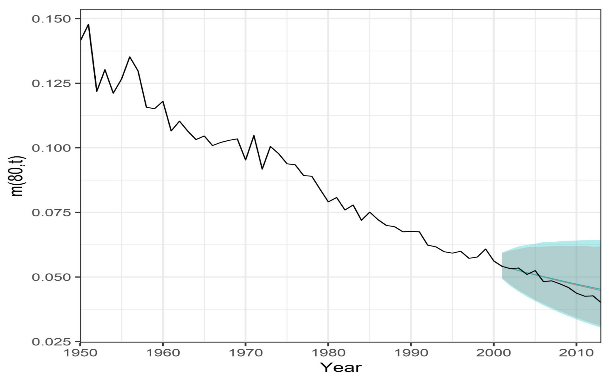
(a) $m(50,t)$ Forecasts



(b) $m(60,t)$ Forecasts



(c) $m(70,t)$ Forecasts



(d) $m(80,t)$ Forecasts

(e) $m(90,t)$ Forecasts

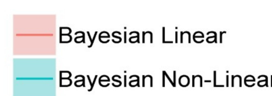
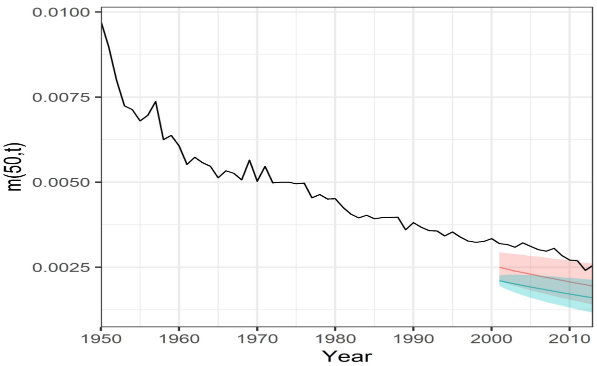
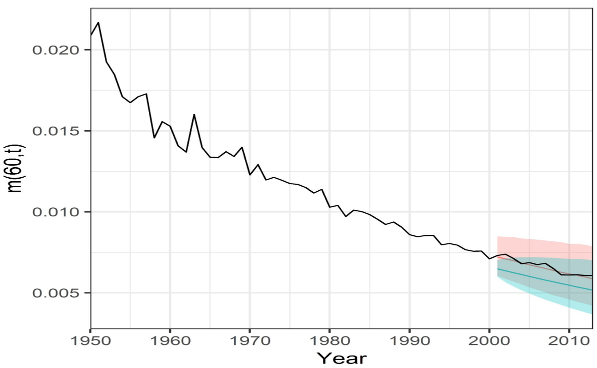


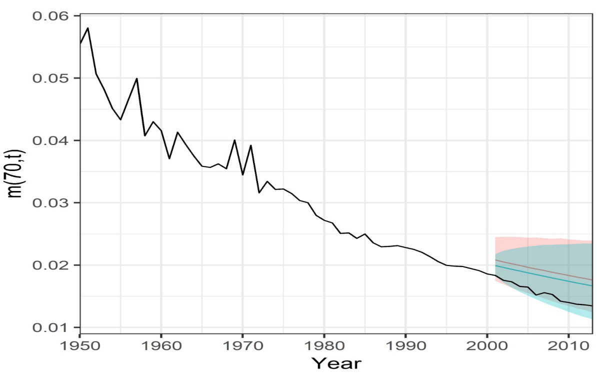
Figure 40: Forecasts over period 2001-2013 for the Spain dataset under the LC model and its variants.



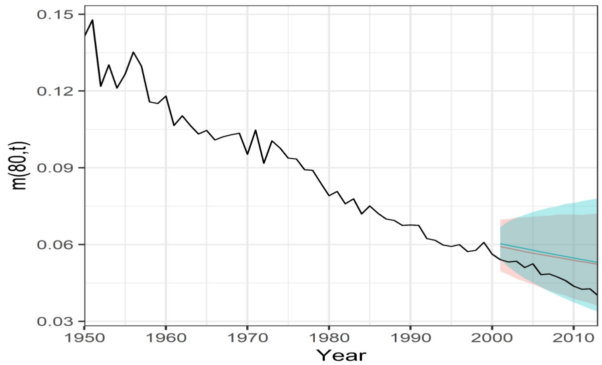
(a) $m(50,t)$ Forecasts



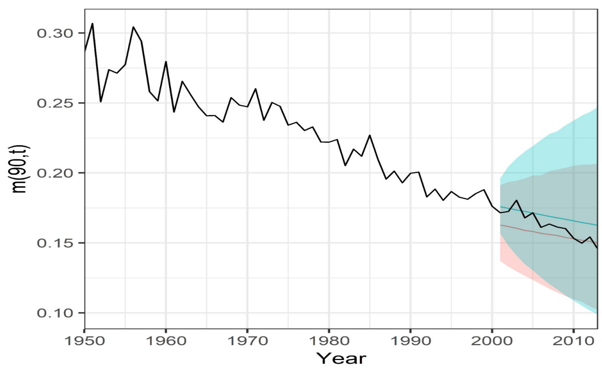
(b) $m(60,t)$ Forecasts



(c) $m(70,t)$ Forecasts



(d) $m(80,t)$ Forecasts



(e) $m(90,t)$ Forecasts

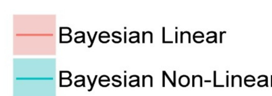
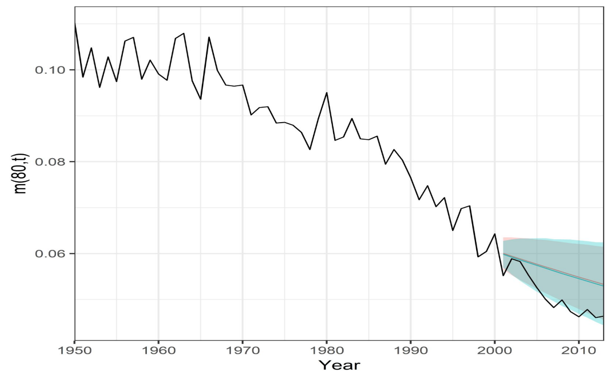


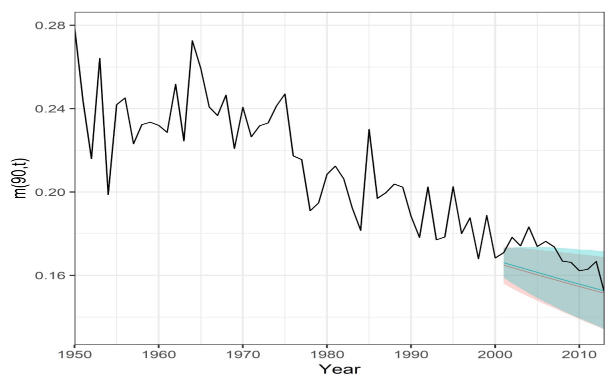
Figure 41: Forecasts over period 2001-2013 for the Spain dataset under the CBD model and its variants.

(a) $m(50,t)$ Forecasts

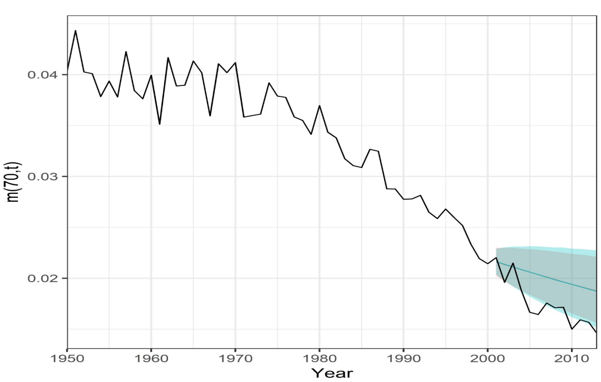


(d) $m(80,t)$ Forecasts

(b) $m(60,t)$ Forecasts



(e) $m(90,t)$ Forecasts



(c) $m(70,t)$ Forecasts

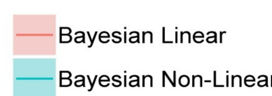
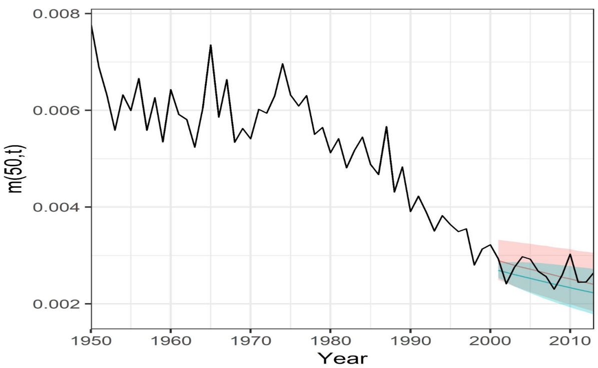
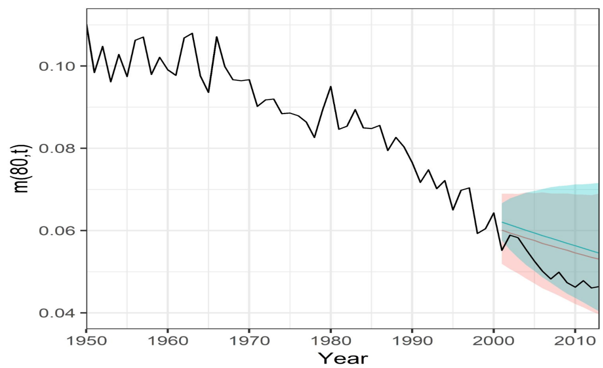


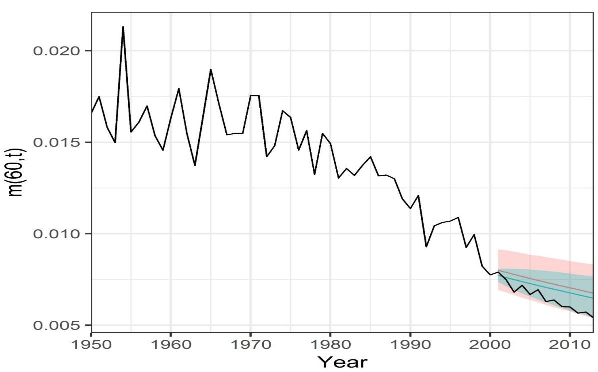
Figure 42: Forecasts over period 2001-2013 for the New Zealand dataset under the LC model and its variants.



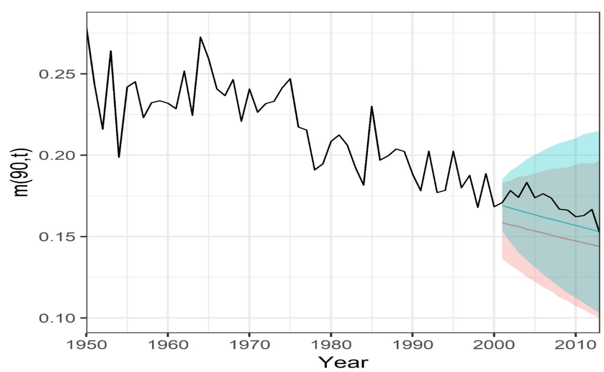
(a) $m(50, t)$ Forecasts



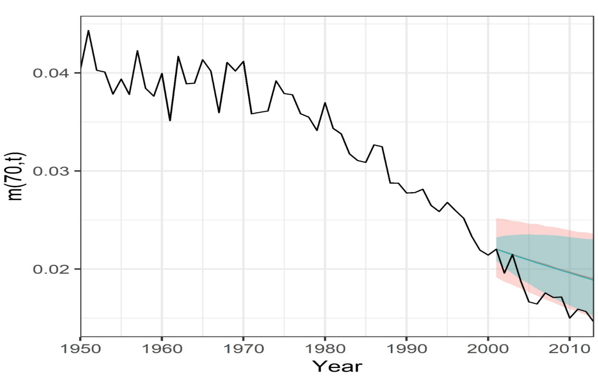
(d) $m(80, t)$ Forecasts



(b) $m(60, t)$ Forecasts



(e) $m(90, t)$ Forecasts



(c) $m(70, t)$ Forecasts

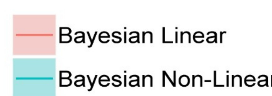
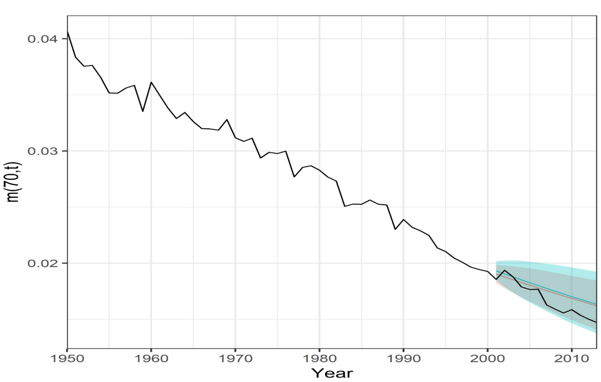


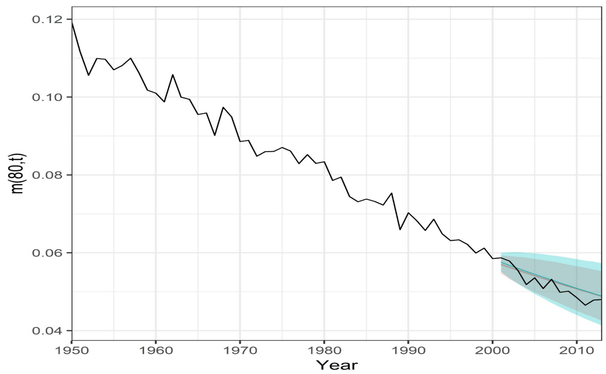
Figure 43: Forecasts over period 2001-2013 for the New Zealand dataset under the CBD model and its variants.

(a) $m(50, t)$ Forecasts

(b) $m(60, t)$ Forecasts



(c) $m(70, t)$ Forecasts



(d) $m(80, t)$ Forecasts

(e) $m(90, t)$ Forecasts

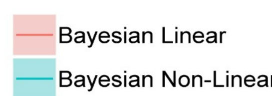
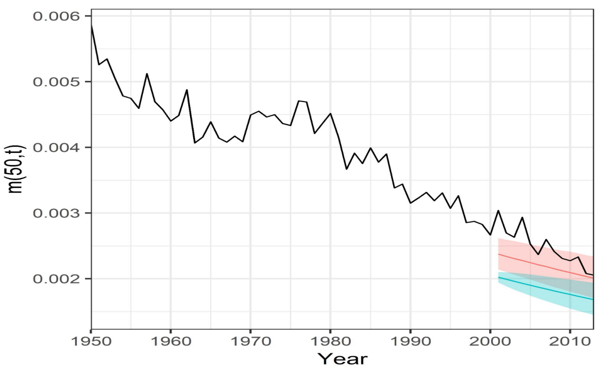
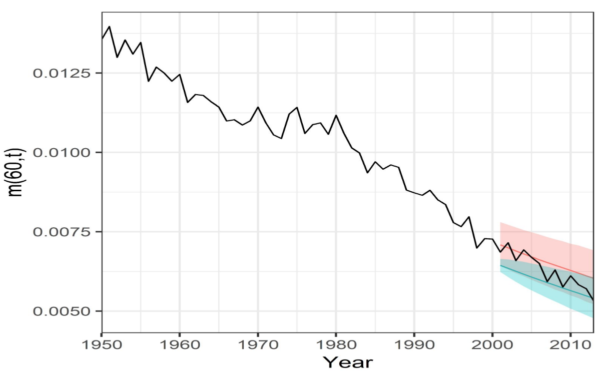


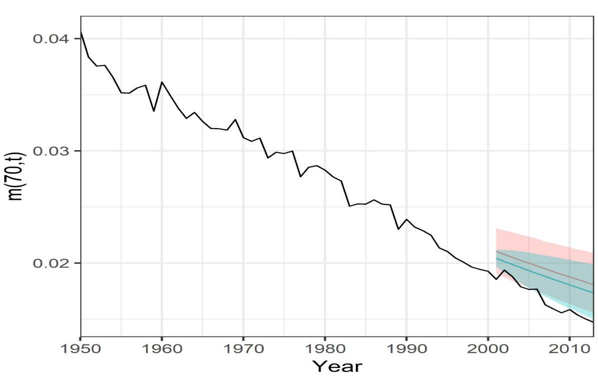
Figure 44: Forecasts over period 2001-2013 for the Sweden dataset under the LC model and its variants.



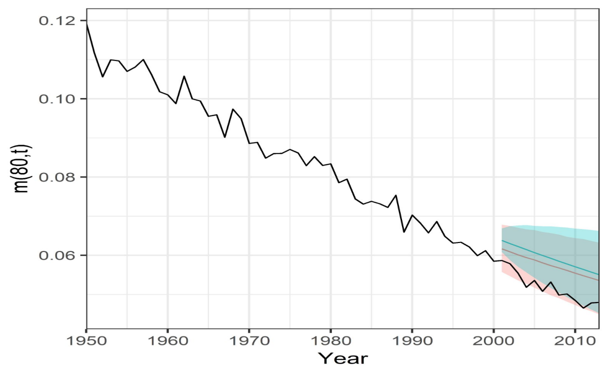
(a) $m(50,t)$ Forecasts



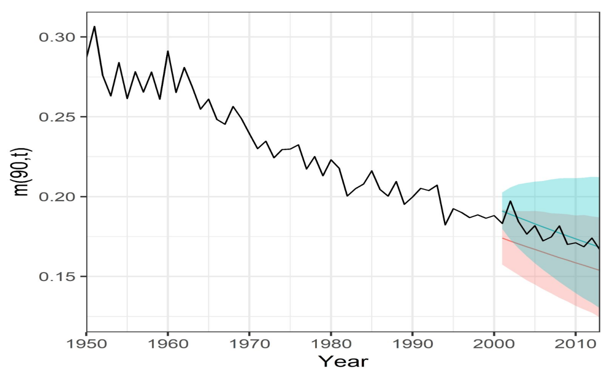
(b) $m(60,t)$ Forecasts



(c) $m(70,t)$ Forecasts



(d) $m(80,t)$ Forecasts



(e) $m(90,t)$ Forecasts

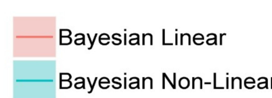
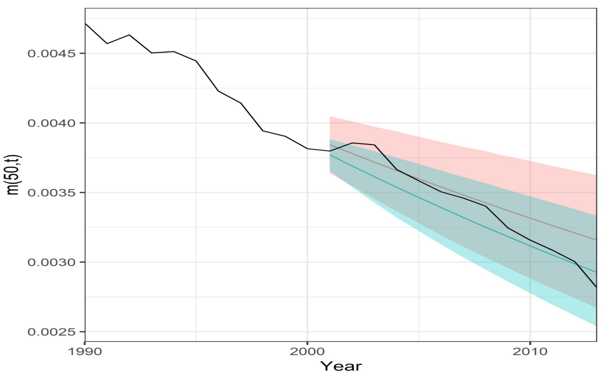
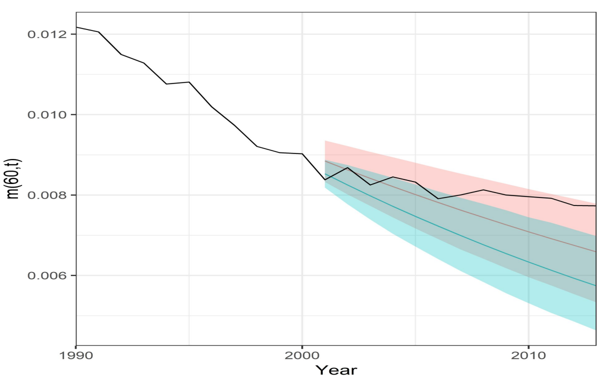


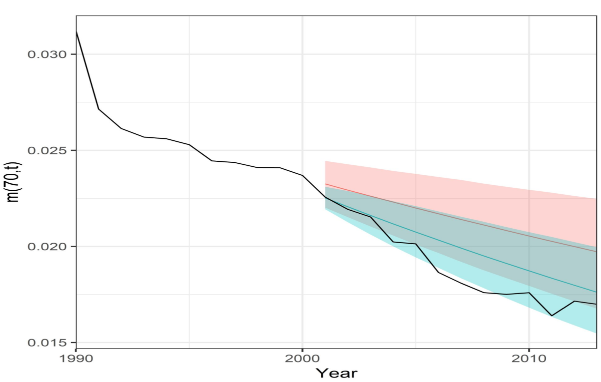
Figure 45: Forecasts over period 2001-2013 for the Sweden dataset under the CBD model and its variants.



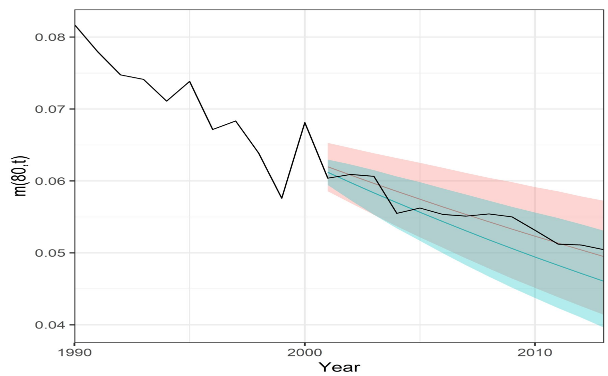
(a) $m(50,t)$ Forecasts



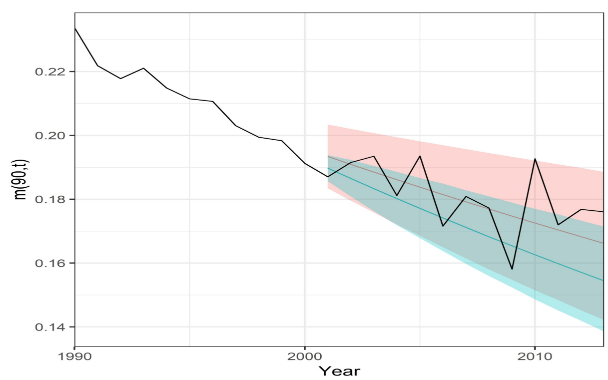
(b) $m(60,t)$ Forecasts



(c) $m(70,t)$ Forecasts



(d) $m(80,t)$ Forecasts



(e) $m(90,t)$ Forecasts

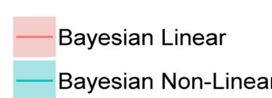
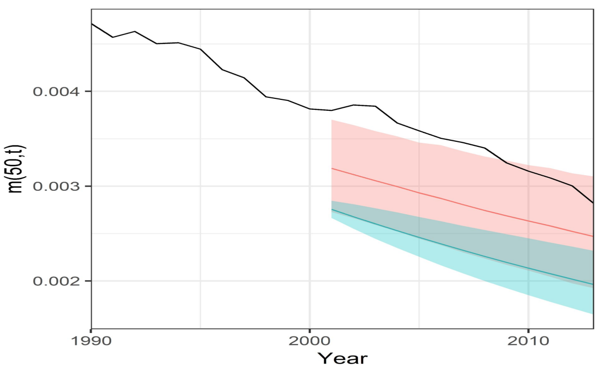
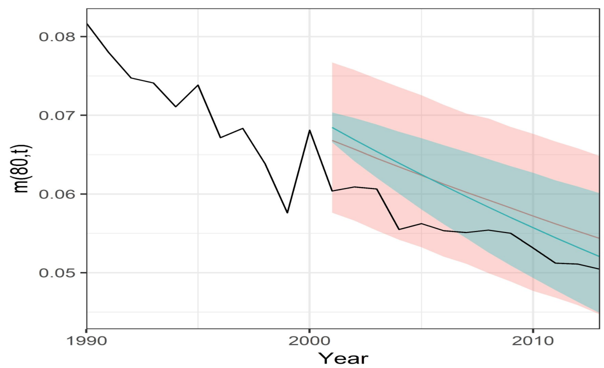


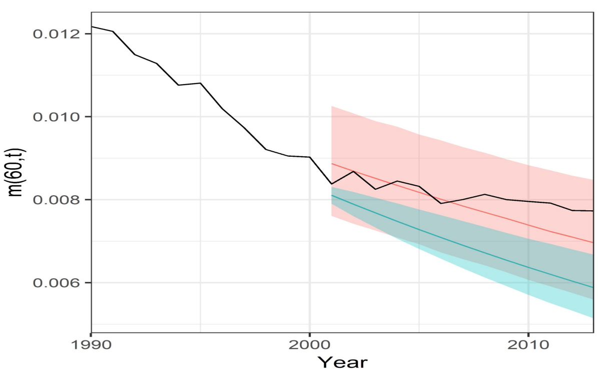
Figure 46: Forecasts over period 2001-2013 for the Germany dataset under the LC model and its variants.



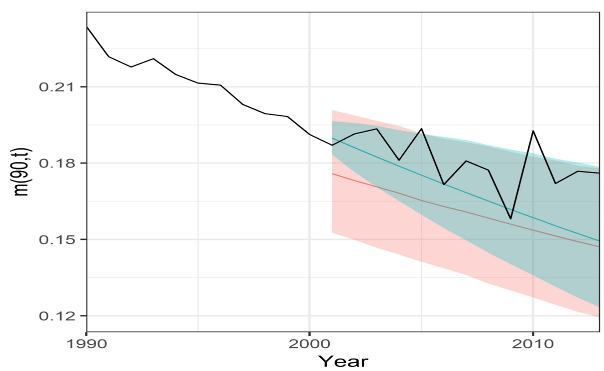
(a) $m(50,t)$ Forecasts



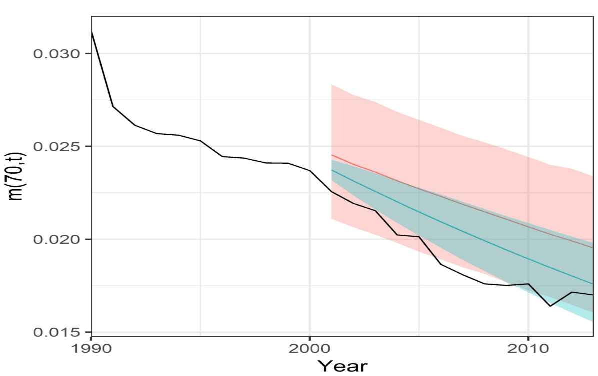
(d) $m(80,t)$ Forecasts



(b) $m(60,t)$ Forecasts



(e) $m(90,t)$ Forecasts



(c) $m(70,t)$ Forecasts

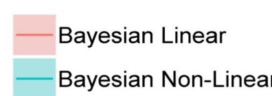
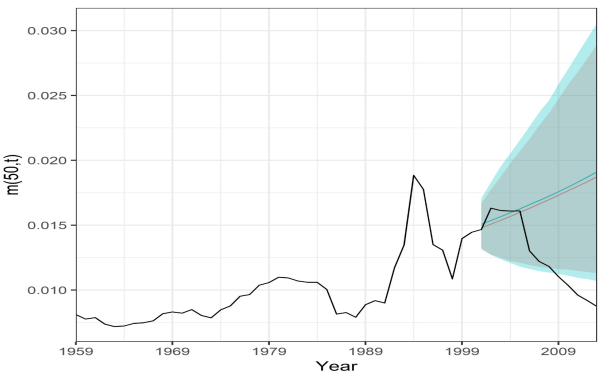
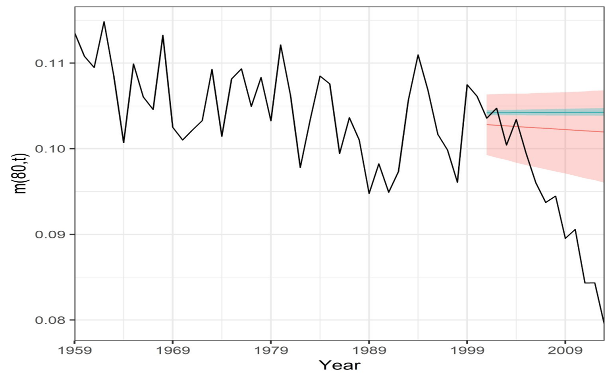


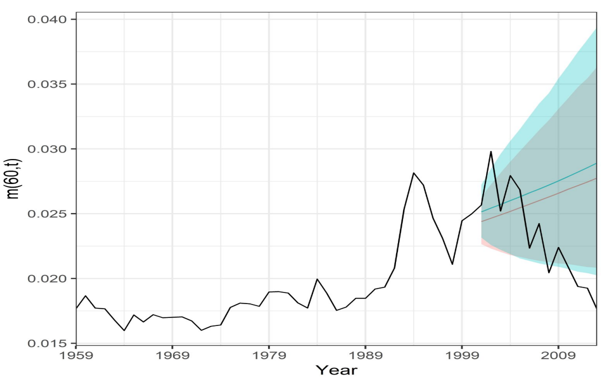
Figure 47: Forecasts over period 2001-2013 for the Germany dataset under the CBD model and its variants.



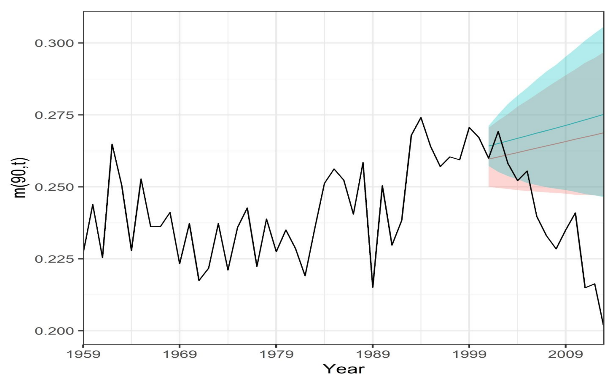
(a) $m(50, t)$ Forecasts



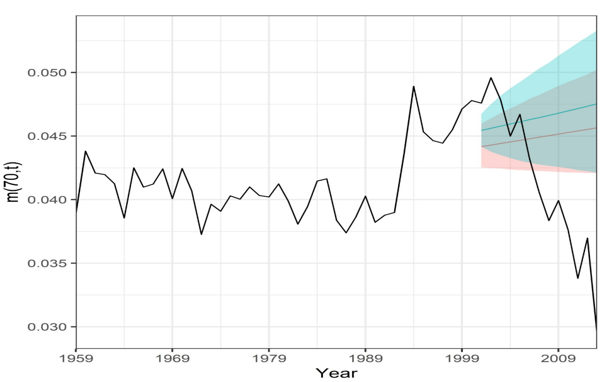
(d) $m(80, t)$ Forecasts



(b) $m(60, t)$ Forecasts



(e) $m(90, t)$ Forecasts



(c) $m(70, t)$ Forecasts

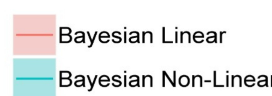
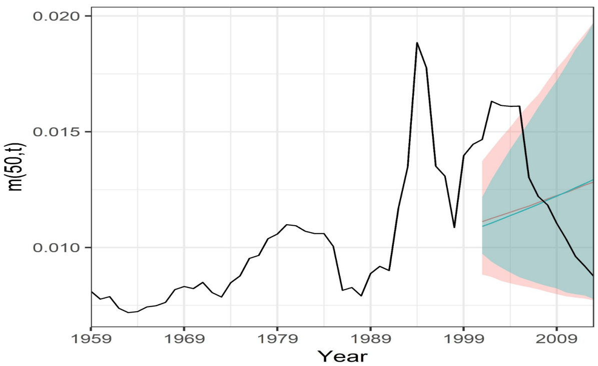
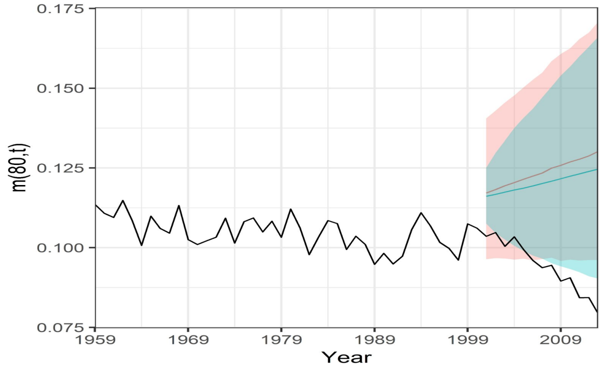


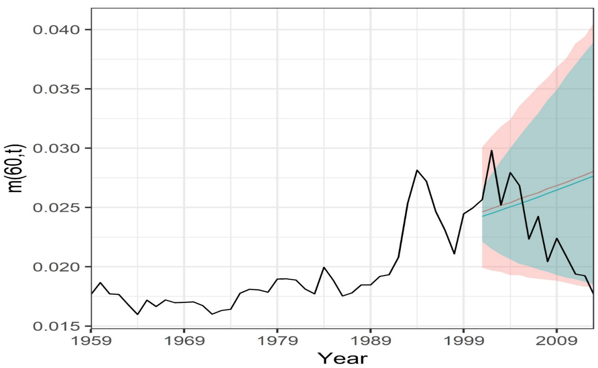
Figure 48: Forecasts over period 2001-2013 for the Russia dataset under the LC model and its variants.



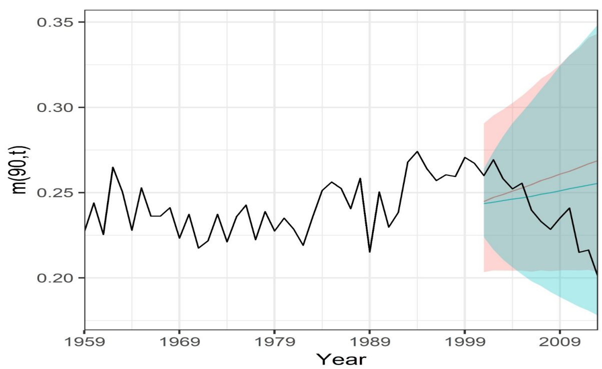
(a) $m(50, t)$ Forecasts



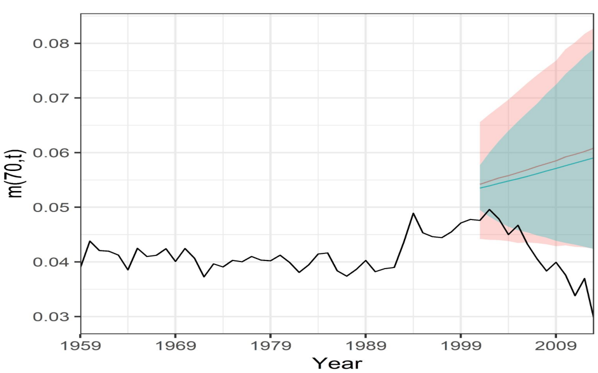
(d) $m(80, t)$ Forecasts



(b) $m(60, t)$ Forecasts



(e) $m(90, t)$ Forecasts



(c) $m(70, t)$ Forecasts

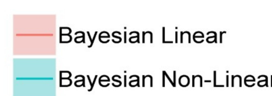


Figure 49: Forecasts over period 2001-2013 for the Russia dataset under the CBD model and its variants.

2. Bayesian annuity Value-At-Risk back-testing Table of Results

LC model: Linear variant

Australia

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9793	10.0162	9.9828	0.3702	0.0334	0
51	9.9577	9.9982	9.9623	0.4064	0.0358	0
52	9.9333	9.9776	9.9488	0.4463	0.0288	0
53	9.9064	9.9548	9.9250	0.4889	0.0298	0
54	9.8770	9.9299	9.8884	0.5355	0.0415	0
55	9.8429	9.9004	9.8740	0.5847	0.0265	0
56	9.8070	9.8702	9.8293	0.6440	0.0409	0
57	9.7658	9.8344	9.7914	0.7026	0.0430	0
58	9.7225	9.7972	9.7652	0.7682	0.0320	0
59	9.6737	9.7548	9.7131	0.8385	0.0418	0
60	9.6211	9.7085	9.6702	0.9086	0.0383	0
61	9.5626	9.6573	9.6146	0.9909	0.0427	0
62	9.5002	9.6035	9.5690	1.0871	0.0345	0
63	9.4312	9.5431	9.5031	1.1863	0.0400	0
64	9.3562	9.4768	9.4336	1.2891	0.0432	0
65	9.2715	9.4017	9.3584	1.4045	0.0434	0
66	9.1787	9.3188	9.2779	1.5266	0.0409	0
67	9.0821	9.2335	9.1884	1.6674	0.0451	0
68	8.9768	9.1392	9.0898	1.8089	0.0494	0
69	8.8575	9.0308	8.9722	1.9565	0.0586	0
70	8.7294	8.9145	8.8291	2.1209	0.0854	0
71	8.5892	8.7852	8.6899	2.2829	0.0953	0
72	8.4406	8.6483	8.5228	2.4613	0.1255	0
73	8.2765	8.4955	8.3340	2.6459	0.1615	0
74	8.0946	8.3239	8.1954	2.8336	0.1285	0
75	7.8994	8.1376	7.9943	3.0152	0.1433	0
76	7.6882	7.9351	7.7661	3.2118	0.1690	0
77	7.4619	7.7168	7.5358	3.4155	0.1810	0
78	7.2311	7.4937	7.3000	3.6317	0.1937	0
79	6.9783	7.2429	7.0113	3.7922	0.2315	0
80	6.7177	6.9857	6.7351	3.9891	0.2506	0
81	6.4534	6.7231	6.4079	4.1783	0.3151	0
82	6.1871	6.4558	6.1584	4.3428	0.2974	0
83	5.9112	6.1749	5.8415	4.4613	0.3334	0
84	5.6306	5.8864	5.5351	4.5428	0.3513	0
85	5.2720	5.5014	5.1972	4.3502	0.3042	0
86	4.9232	5.1259	4.7748	4.1183	0.3511	0
87	4.5753	4.7515	4.4311	3.8505	0.3204	0
88	4.2395	4.3914	4.0895	3.5837	0.3019	0
89	3.8941	4.0243	3.8028	3.3419	0.2215	0
90	3.5553	3.6625	3.4631	3.0166	0.1994	0
91	3.2043	3.2910	3.1181	2.7074	0.1729	0
92	2.8525	2.9212	2.7851	2.4073	0.1361	0
93	2.4721	2.5230	2.4024	2.0600	0.1206	0
94	2.0348	2.0675	1.9787	1.6045	0.0888	0
95	1.5150	1.5315	1.4919	1.0949	0.0396	0

United Kingdom

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9112	9.9595	9.9201	0.4881	0.0395	0
51	9.8811	9.9341	9.8976	0.5366	0.0365	0
52	9.8477	9.9058	9.8747	0.5903	0.0311	0
53	9.8103	9.8740	9.8426	0.6485	0.0314	0
54	9.7691	9.8383	9.8147	0.7084	0.0235	0
55	9.7231	9.7987	9.7836	0.7771	0.0150	0
56	9.6735	9.7555	9.7344	0.8477	0.0212	0
57	9.6182	9.7069	9.6814	0.9225	0.0255	0
58	9.5587	9.6545	9.6374	1.0021	0.0171	0
59	9.4919	9.5955	9.5830	1.0923	0.0125	0
60	9.4197	9.5312	9.5279	1.1834	0.0034	0
61	9.3433	9.4636	9.4571	1.2880	0.0065	0
62	9.2611	9.3906	9.3933	1.3976	-0.0027	1
63	9.1703	9.3091	9.3217	1.5136	-0.0126	1
64	9.0722	9.2213	9.2490	1.6438	-0.0277	1
65	8.9684	9.1286	9.1546	1.7866	-0.0260	1
66	8.8588	9.0315	9.0497	1.9488	-0.0182	1
67	8.7437	8.9296	8.9448	2.1263	-0.0152	1
68	8.6200	8.8195	8.8258	2.3145	-0.0063	1
69	8.4866	8.6995	8.6810	2.5080	0.0185	0
70	8.3445	8.5706	8.5145	2.7091	0.0560	0
71	8.1957	8.4379	8.3569	2.9550	0.0810	0
72	8.0397	8.2970	8.1704	3.2010	0.1266	0
73	7.8711	8.1428	7.9836	3.4521	0.1592	0
74	7.6926	7.9792	7.7734	3.7254	0.2058	0
75	7.5039	7.8042	7.5462	4.0006	0.2579	0
76	7.3020	7.6142	7.3473	4.2755	0.2670	0
77	7.0883	7.4121	7.1042	4.5676	0.3080	0
78	6.8653	7.1975	6.8808	4.8381	0.3167	0
79	6.6283	6.9669	6.5866	5.1090	0.3803	0
80	6.3795	6.7210	6.3829	5.3537	0.3381	0
81	6.1340	6.4769	5.9044	5.5906	0.5725	0
82	5.8792	6.2201	5.8909	5.7985	0.3292	0
83	5.6191	5.9554	5.5380	5.9857	0.4174	0
84	5.3559	5.6805	5.1813	6.0618	0.4992	0
85	5.0261	5.3171	4.8808	5.7895	0.4363	0
86	4.7042	4.9605	4.5390	5.4469	0.4215	0
87	4.3842	4.6079	4.2364	5.1027	0.3715	0
88	4.0667	4.2577	3.9169	4.6968	0.3408	0
89	3.7493	3.9118	3.6123	4.3346	0.2995	0
90	3.4294	3.5624	3.3223	3.8785	0.2400	0
91	3.1068	3.2133	2.9890	3.4274	0.2243	0
92	2.7679	2.8502	2.6874	2.9727	0.1628	0
93	2.4016	2.4593	2.3306	2.4038	0.1287	0
94	1.9879	2.0239	1.9462	1.8104	0.0776	0
95	1.4896	1.5078	1.4661	1.2242	0.0417	0

Italy

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9538	10.0024	9.9729	0.4880	0.0294	0
51	9.9294	9.9822	9.9500	0.5322	0.0322	0
52	9.9023	9.9597	9.9312	0.5795	0.0285	0
53	9.8730	9.9355	9.9010	0.6328	0.0345	0
54	9.8408	9.9087	9.8723	0.6896	0.0364	0
55	9.8056	9.8793	9.8417	0.7511	0.0376	0
56	9.7677	9.8479	9.8096	0.8204	0.0383	0
57	9.7262	9.8134	9.7678	0.8967	0.0456	0
58	9.6806	9.7752	9.7380	0.9775	0.0372	0
59	9.6311	9.7340	9.6961	1.0683	0.0378	0
60	9.5769	9.6892	9.6642	1.1718	0.0250	0
61	9.5186	9.6413	9.6128	1.2892	0.0285	0
62	9.4567	9.5907	9.5561	1.4167	0.0346	0
63	9.3888	9.5347	9.4888	1.5539	0.0459	0
64	9.3146	9.4735	9.4250	1.7054	0.0484	0
65	9.2352	9.4092	9.3367	1.8837	0.0725	0
66	9.1483	9.3390	9.2515	2.0850	0.0875	0
67	9.0534	9.2615	9.1424	2.2985	0.1191	0
68	8.9503	9.1776	9.0484	2.5403	0.1292	0
69	8.8367	9.0844	8.9299	2.8031	0.1546	0
70	8.7120	8.9817	8.7963	3.0959	0.1854	0
71	8.5746	8.8657	8.6425	3.3946	0.2232	0
72	8.4252	8.7397	8.4846	3.7330	0.2551	0
73	8.2620	8.5997	8.2908	4.0884	0.3090	0
74	8.0832	8.4440	8.1062	4.4630	0.3378	0
75	7.8893	8.2731	7.9058	4.8638	0.3673	0
76	7.6790	8.0833	7.6726	5.2643	0.4106	0
77	7.4545	7.8763	7.4345	5.6584	0.4418	0
78	7.2135	7.6529	7.2051	6.0905	0.4478	0
79	6.9594	7.4084	6.9324	6.4520	0.4760	0
80	6.6936	7.1484	6.6654	6.7955	0.4830	0
81	6.4150	6.8731	6.3386	7.1415	0.5345	0
82	6.1264	6.5823	6.0542	7.4417	0.5281	0
83	5.8300	6.2765	5.7726	7.6583	0.5039	0
84	5.5336	5.9650	5.4799	7.7956	0.4850	0
85	5.1689	5.5553	5.1253	7.4744	0.4299	0
86	4.8133	5.1518	4.7808	7.0338	0.3710	0
87	4.4678	4.7627	4.4134	6.6027	0.3493	0
88	4.1300	4.3806	4.0915	6.0665	0.2891	0
89	3.7980	4.0094	3.7350	5.5666	0.2744	0
90	3.4660	3.6397	3.4317	5.0126	0.2081	0
91	3.1335	3.2700	3.0758	4.3573	0.1942	0
92	2.7865	2.8904	2.7524	3.7314	0.1380	0
93	2.4156	2.4872	2.3592	2.9633	0.1280	0
94	1.9991	2.0441	1.9685	2.2519	0.0757	0
95	1.4954	1.5175	1.4779	1.4791	0.0397	0

France

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.8846	9.9335	9.8781	0.4941	0.0554	0
51	9.8622	9.9151	9.8599	0.5361	0.0551	0
52	9.8380	9.8952	9.8402	0.5811	0.0550	0
53	9.8126	9.8743	9.8187	0.6291	0.0556	0
54	9.7846	9.8510	9.7931	0.6785	0.0580	0
55	9.7550	9.8269	9.7676	0.7368	0.0593	0
56	9.7237	9.8014	9.7375	0.7984	0.0638	0
57	9.6890	9.7728	9.7088	0.8652	0.0640	0
58	9.6519	9.7426	9.6780	0.9395	0.0646	0
59	9.6116	9.7095	9.6536	1.0193	0.0560	0
60	9.5684	9.6747	9.6088	1.1116	0.0660	0
61	9.5225	9.6381	9.5763	1.2138	0.0617	0
62	9.4722	9.5973	9.5219	1.3209	0.0754	0
63	9.4177	9.5537	9.4744	1.4436	0.0793	0
64	9.3581	9.5064	9.4181	1.5847	0.0882	0
65	9.2919	9.4534	9.3506	1.7381	0.1028	0
66	9.2188	9.3946	9.2797	1.9073	0.1149	0
67	9.1373	9.3294	9.1884	2.1024	0.1410	0
68	9.0465	9.2557	9.1006	2.3116	0.1550	0
69	8.9457	9.1726	9.0024	2.5365	0.1702	0
70	8.8319	9.0778	8.8893	2.7842	0.1885	0
71	8.7055	8.9718	8.7572	3.0592	0.2146	0
72	8.5637	8.8511	8.6228	3.3559	0.2284	0
73	8.4068	8.7146	8.4586	3.6623	0.2560	0
74	8.2326	8.5610	8.2789	3.9890	0.2821	0
75	8.0407	8.3888	8.0958	4.3291	0.2930	0
76	7.8305	8.1962	7.8918	4.6706	0.3044	0
77	7.6016	7.9836	7.6560	5.0244	0.3275	0
78	7.3561	7.7504	7.4276	5.3602	0.3229	0
79	7.0953	7.5004	7.1598	5.7098	0.3407	0
80	6.8186	7.2283	6.8742	6.0081	0.3540	0
81	6.5330	6.9435	6.5666	6.2829	0.3769	0
82	6.2389	6.6466	6.2816	6.5342	0.3649	0
83	5.9382	6.3377	5.9993	6.7270	0.3383	0
84	5.6362	6.0238	5.6693	6.8768	0.3545	0
85	5.2637	5.6087	5.2436	6.5543	0.3651	0
86	4.8994	5.2017	4.9162	6.1710	0.2855	0
87	4.5440	4.8064	4.5486	5.7746	0.2577	0
88	4.1949	4.4189	4.1859	5.3385	0.2329	0
89	3.8540	4.0420	3.8349	4.8788	0.2071	0
90	3.5142	3.6670	3.5028	4.3483	0.1642	0
91	3.1705	3.2908	3.1149	3.7946	0.1759	0
92	2.8164	2.9065	2.7836	3.1996	0.1229	0
93	2.4344	2.4956	2.3983	2.5146	0.0974	0
94	2.0068	2.0444	1.9928	1.8734	0.0516	0
95	1.4992	1.5175	1.4912	1.2202	0.0263	0

Spain

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9450	9.9986	9.9400	0.5392	0.0586	0
51	9.9234	9.9816	9.9207	0.5865	0.0609	0
52	9.9017	9.9658	9.8990	0.6476	0.0668	0
53	9.8761	9.9460	9.8751	0.7078	0.0709	0
54	9.8498	9.9262	9.8523	0.7760	0.0739	0
55	9.8201	9.9037	9.8218	0.8514	0.0819	0
56	9.7882	9.8800	9.7985	0.9374	0.0815	0
57	9.7526	9.8533	9.7621	1.0328	0.0912	0
58	9.7144	9.8248	9.7299	1.1367	0.0949	0
59	9.6709	9.7920	9.6928	1.2522	0.0992	0
60	9.6257	9.7589	9.6509	1.3837	0.1080	0
61	9.5725	9.7185	9.6028	1.5252	0.1157	0
62	9.5190	9.6802	9.5457	1.6937	0.1345	0
63	9.4550	9.6310	9.4750	1.8621	0.1560	0
64	9.3847	9.5775	9.4182	2.0546	0.1594	0
65	9.3070	9.5180	9.3565	2.2676	0.1615	0
66	9.2188	9.4498	9.2639	2.5059	0.1859	0
67	9.1224	9.3751	9.1758	2.7705	0.1993	0
68	9.0142	9.2898	9.0714	3.0570	0.2183	0
69	8.8915	9.1902	8.9503	3.3588	0.2398	0
70	8.7614	9.0853	8.8187	3.6967	0.2665	0
71	8.6055	8.9493	8.6776	3.9950	0.2717	0
72	8.4456	8.8146	8.5102	4.3684	0.3044	0
73	8.2587	8.6475	8.3291	4.7076	0.3184	0
74	8.0597	8.4676	8.1320	5.0613	0.3357	0
75	7.8420	8.2662	7.9207	5.4100	0.3456	0
76	7.6060	8.0425	7.7051	5.7390	0.3373	0
77	7.3560	7.8019	7.4577	6.0616	0.3442	0
78	7.0933	7.5444	7.2083	6.3595	0.3362	0
79	6.8040	7.2474	6.9338	6.5175	0.3137	0
80	6.5219	6.9581	6.6326	6.6881	0.3255	0
81	6.2067	6.6244	6.3381	6.7299	0.2863	0
82	5.9188	6.3222	6.0094	6.8150	0.3128	0
83	5.6032	5.9746	5.7310	6.6271	0.2436	0
84	5.2991	5.6391	5.4276	6.4165	0.2115	0
85	4.9331	5.2206	5.0418	5.8269	0.1787	0
86	4.5819	4.8226	4.7064	5.2545	0.1163	0
87	4.2376	4.4360	4.3602	4.6804	0.0758	0
88	3.9113	4.0695	4.0519	4.0458	0.0177	0
89	3.5826	3.7060	3.7089	3.4448	-0.0029	1
90	3.2742	3.3711	3.3755	2.9596	-0.0044	1
91	2.9392	3.0074	3.0591	2.3229	-0.0517	1
92	2.6609	2.7173	2.7038	2.1186	0.0135	0
93	2.3156	2.3541	2.3781	1.6598	-0.0240	1
94	1.9312	1.9572	1.9824	1.3483	-0.0252	1
95	1.4637	1.4780	1.4751	0.9789	0.0029	0

LC model: Linear variant

x	New Zealand $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	Sweden $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	Germany $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	Russia $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9211	9.9710	9.9513	0.5031	0.0197	0	9.9415	9.9692	9.9639	0.2786	0.0053	0	9.9103	9.9422	9.8861	0.3217	0.0561	0	9.1280	9.4311	9.2446	3.3207	0.1865	0
51	9.8938	9.9480	9.9191	0.5486	0.0289	0	9.9179	9.9482	9.9453	0.3057	0.0029	0	9.8889	9.9245	9.8616	0.3599	0.0629	0	9.0827	9.3889	9.1729	3.3714	0.2160	0
52	9.8635	9.9232	9.9042	0.6050	0.0190	0	9.8924	9.9254	9.9227	0.3339	0.0027	0	9.8653	9.9049	9.8363	0.4011	0.0687	0	9.0307	9.3393	9.0677	3.4172	0.2716	0
53	9.8312	9.8963	9.8706	0.6623	0.0257	0	9.8642	9.9006	9.8928	0.3687	0.0078	0	9.8391	9.8830	9.8068	0.4461	0.0761	0	8.9704	9.2839	9.1476	3.4942	0.1363	0
54	9.7951	9.8659	9.8562	0.7227	0.0097	0	9.8330	9.8725	9.8679	0.4020	0.0045	0	9.8082	9.8566	9.7731	0.4933	0.0835	0	8.9094	9.2257	8.9133	3.5497	0.3124	0
55	9.7581	9.8354	9.8333	0.7924	0.0021	0	9.7990	9.8425	9.8361	0.4438	0.0064	0	9.7726	9.8251	9.7488	0.5365	0.0763	0	8.8434	9.1618	9.0015	3.6001	0.1603	0
56	9.7202	9.8026	9.7846	0.8483	0.0181	0	9.7626	9.8102	9.7988	0.4879	0.0114	0	9.7306	9.7865	9.6991	0.5740	0.0873	0	8.7785	9.0973	8.8733	3.6325	0.2240	0
57	9.6746	9.7641	9.7223	0.9246	0.0418	0	9.7217	9.7740	9.7514	0.5382	0.0226	0	9.6826	9.7416	9.6588	0.6092	0.0827	0	8.7053	9.0247	8.8001	3.6692	0.2246	0
58	9.6246	9.7211	9.6985	1.0025	0.0226	0	9.6776	9.7353	9.7172	0.5961	0.0181	0	9.6291	9.6909	9.6370	0.6422	0.0539	0	8.6273	8.9453	8.8848	3.6870	0.0605	0
59	9.5688	9.6720	9.6489	1.0780	0.0231	0	9.6291	9.6927	9.6675	0.6604	0.0252	0	9.5690	9.6329	9.5802	0.6685	0.0527	0	8.5453	8.8605	8.5613	3.6886	0.2991	0
60	9.5103	9.6207	9.5913	1.1606	0.0294	0	9.5756	9.6455	9.6264	0.7301	0.0192	0	9.5034	9.5696	9.5597	0.6972	0.0099	0	8.4577	8.7677	8.6315	3.6654	0.1362	0
61	9.4487	9.5682	9.5364	1.2654	0.0318	0	9.5178	9.5948	9.5637	0.8091	0.0311	0	9.4341	9.5029	9.4789	0.7300	0.0240	0	8.3705	8.6741	8.5191	3.6274	0.1550	0
62	9.3816	9.5108	9.4576	1.3774	0.0532	0	9.4540	9.5390	9.5224	0.8986	0.0165	0	9.3603	9.4319	9.4277	0.7641	0.0041	0	8.2812	8.5712	8.3686	3.5016	0.2026	0
63	9.3073	9.4462	9.4041	1.4916	0.0420	0	9.3849	9.4788	9.4527	1.0001	0.0261	0	9.2824	9.3581	9.3652	0.8158	-0.0071	1	8.1802	8.4595	8.3787	3.4137	0.0808	0
64	9.2271	9.3770	9.3320	1.6240	0.0450	0	9.3084	9.4117	9.3851	1.1102	0.0267	0	9.2004	9.2798	9.2773	0.8634	0.0025	0	8.0766	8.3411	8.1978	3.2759	0.1433	0
65	9.1410	9.3022	9.2379	1.7636	0.0643	0	9.2267	9.3403	9.2898	1.2315	0.0505	0	9.1158	9.2020	9.1905	0.9456	0.0114	0	7.9654	8.2131	8.0594	3.1094	0.1537	0
66	9.0453	9.2166	9.1430	1.8948	0.0736	0	9.1365	9.2619	9.1905	1.3728	0.0714	0	9.0295	9.1247	9.0496	1.0541	0.0751	0	7.8464	8.0773	7.9169	2.9430	0.1604	0
67	8.9453	9.1304	9.0443	2.0694	0.0861	0	9.0343	9.1710	9.1074	1.5140	0.0637	0	8.9349	9.0409	8.9766	1.1853	0.0643	0	7.7207	7.9326	7.8550	2.7448	0.0776	0
68	8.8357	9.0344	8.9082	2.2492	0.1263	0	8.9212	9.0700	8.9778	1.6682	0.0922	0	8.8358	8.9540	8.8504	1.3376	0.1036	0	7.5793	7.7753	7.6654	2.5863	0.1099	0
69	8.7129	8.9249	8.8277	2.4323	0.0972	0	8.7959	8.9578	8.8479	1.8400	0.1098	0	8.7285	8.8600	8.7027	1.5056	0.1572	0	7.4276	7.6083	7.4584	2.4325	0.1499	0
70	8.5798	8.8053	8.6591	2.6283	0.1462	0	8.6581	8.8337	8.7466	2.0271	0.0870	0	8.6081	8.7534	8.5501	1.6884	0.2034	0	7.2657	7.4325	7.3445	2.2951	0.0880	0
71	8.4341	8.6732	8.5435	2.8353	0.1297	0	8.5048	8.6934	8.5814	2.2178	0.1120	0	8.4715	8.6299	8.4011	1.8690	0.2287	0	7.0840	7.2458	7.1416	2.2842	0.1042	0
72	8.2845	8.5370	8.3845	3.0475	0.1524	0	8.3413	8.5430	8.4248	2.4184	0.1183	0	8.3214	8.4941	8.2249	2.0758	0.2693	0	6.9015	7.0505	6.9569	2.1589	0.0936	0
73	8.1129	8.3768	8.1917	3.2537	0.1851	0	8.1607	8.3765	8.2555	2.6440	0.1210	0	8.1564	8.3432	8.0629	2.2904	0.2803	0	6.6929	6.8435	6.7232	2.2489	0.1203	0
74	7.9282	8.2040	8.0180	3.4785	0.1860	0	7.9662	8.1949	8.0369	2.8713	0.1580	0	7.9665	8.1660	7.8924	2.5043	0.2736	0	6.4703	6.6234	6.5745	2.3663	0.0488	0
75	7.7270	8.0110	7.7942	3.6754	0.2168	0	7.7588	8.0017	7.8595	3.1304	0.1422	0	7.7640	7.9762	7.6943	2.7338	0.2820	0	6.2404	6.3967	6.3831	2.5046	0.0136	0
76	7.5164	7.8113	7.5782	3.9225	0.2331	0	7.5332	7.7861	7.5841	3.3571	0.2020	0	7.5428	7.7657	7.4687	2.9556	0.2970	0	5.9976	6.1616	6.1887	2.7350	-0.0271	1
77	7.2852	7.5862	7.3687	4.1318	0.2176	0	7.2932	7.5557	7.3422	3.5990	0.2135	0	7.3012	7.5330	7.2441	3.1749	0.2889	0	5.7489	5.9224	5.8660	3.0184	0.0564	0
78	7.0471	7.3583	7.1080	4.4158	0.2503	0	7.0397	7.3089	7.0660	3.8241	0.2429	0	7.0480	7.2877	6.9665	3.4005	0.3212	0	5.4941	5.6759	5.8095	3.3081	-0.1337	1
79	6.7854	7.1019	6.7526	4.6646	0.3493	0	6.7995	7.0751	6.7528	4.0534	0.3223	0	6.7758	7.0197	6.6449	3.6003	0.3748	0	5.2224	5.4238	5.4582	3.8562	-0.0345	1
80	6.5783	6.8922	6.5558	4.7713	0.3364	0	6.5228	6.7980	6.4898	4.2201	0.3082	0	6.4916	6.7353	6.3747	3.7534	0.3606	0	4.9688	5.1785	5.1802	4.2202	-0.0017	1
81	6.3320	6.6505	6.1917	5.0311	0.4589	0	6.2345	6.5065	6.1743	4.3633	0.3323	0	6.1974	6.4376	5.8474	3.8758	0.5902	0	4.7058	4.9417	4.8733	5.0131	0.0684	0
82	6.0773	6.4004	5.9698	5.3161	0.4306	0	5.9401	6.2060	5.8939	4.4766	0.3121	0	5.8797	6.1096	6.0839	3.9113	0.0257	0	4.4653	4.7092	4.6195	5.4635	0.0898	0
83	5.8191	6.1385	5.6619	5.4876	0.4766	0	5.6384	5.8948	5.5279	4.5474	0.3669	0	5.5675	5.7887	5.5173	3.9718	0.2714	0	4.2144	4.4841	4.5215	6.4005	-0.0374	1
84	5.5498	5.8617	5.3041	5.6206	0.5577	0	5.3350	5.5801	5.2461	4.5934	0.3340	0	5.2580	5.4652	5.1217	3.9397	0.3434	0	3.9773	4.2683	4.2097	7.3169	0.0586	0
85	5.1981	5.4774	5.0171	5.3727	0.4603	0	4.9842	5.2026	4.8590	4.3825	0.3436	0	4.8952	5.0766	4.9300	3.7037	0.1465	0	3.7360	4.0279	3.9581	7.8146	0.0698	0
86	4.8642	5.1131	4.5849	5.1158	0.5282	0	4.6425	4.8337	4.4997	4.1186	0.3340	0	4.5520	4.7087	4.3553	3.4437	0.3535	0	3.5026	3.7967	3.6299	8.3955	0.1668	0
87	4.5164	4.7336	4.2225	4.8088	0.5110	0	4.3073	4.4724	4.1952	3.8335	0.2772	0	4.2089	4.3410	4.1629	3.1405	0.1782	0	3.2815	3.5763	3.4169	8.9828	0.1594	0
88	4.1879	4.3780	4.0022	4.5398	0.3758	0	3.9773	4.1182	3.8725	3.5405	0.2456	0	3.8826	3.9936	3.8505	2.8581	0.1431	0	3.0674	3.3570	3.2340	9.4429	0.1230	0
89	3.8536	4.0183	3.6693	4.2745	0.3490	0	3.6567	3.7747	3.5470	3.2279	0.2277	0	3.5632	3.6527	3.5331	2.5124	0.1196	0	2.8583	3.1399	2.9343	9.8511	0.2056	0
90	3.5093	3.6481	3.3595	3.9550	0.2886	0	3.3375	3.4347	3.2705	2.9101	0.1641	0	3.2558	3.3283	3.2417	2.2261	0.0865	0	2.6648	2.9227	2.7585	9.6769	0.1641	0
91	3.1867	3.3025	3.0227	3.6316	0.2798	0	3.0165	3.0939	2.9281	2.5641	0.1658	0	2.9463	3.0022	2.9286	1.8977	0.0736	0	2.4483	2.7024	2.5201	10.3803	0.1823	0
92	2.8522	2.9495	2.6454	3.4106	0.3042	0	2.6895	2.7494	2.6073	2.2284	0.1421	0	2.6298	2.6715	2.6256	1.5853	0.0459	0	2.2569	2.4554	2.2914	8.7943	0.1640	0
93	2.4625	2.5363	2.3352	2.9968	0.2011	0	2.3396	2.3840	2.2668	1.8977	0.1172	0	2.2955	2.3256	2.2867	1.3124	0.0389	0	2.0207	2.1783	2.0314	7.7978	0.1469	0
94	2.0271	2.0772	1.9375	2.4746	0.1397	0	1.9457	1.9753	1.9003	1.5227	0.0749	0	1.9092	1.9288	1.9028	1.0246	0.0259	0	1.7415	1.8465	1.7411	6.0321	0.1054	0
95	1.5022	1.5305	1.4813	1.8806	0.0492	0	1.4617	1.4778	1.4420	1.0975	0.0358	0	1.4460	1.4562	1.4466	0.7046	0.0096	0	1.3666	1.4191	1.3702	3.8467	0.0490	0

LC model: Nonlinear variant

x	Australia						United Kingdom						Italy						France						Spain					
	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9777	10.0180	9.9828	0.4040	0.0352	0	9.9044	9.9602	9.9201	0.5631	0.0401	0	9.9494	10.0013	9.9729	0.5220	0.0284	0	9.8827	9.9334	9.8781	0.5131	0.0553	0	9.9455	10.0036	9.9400	0.5843	0.0636	0
51	9.9559	10.0001	9.9623	0.4436	0.0378	0	9.8735	9.9345	9.8976	0.6180	0.0370	0	9.9245	9.9810	9.9500	0.5691	0.0309	0	9.8600	9.9146	9.8599	0.5539	0.0547	0	9.9235	9.9866	9.9207	0.6359	0.0660	0
52	9.9313	9.9798	9.9488	0.4878	0.0310	0	9.8392	9.9062	9.8747	0.6803	0.0315	0	9.8970	9.9585	9.9312	0.6206	0.0272	0	9.8354	9.8945	9.8402	0.6002	0.0543	0	9.9018	9.9713	9.8990	0.7016	0.0723	0
53	9.9045	9.9572	9.9250	0.5324	0.0322	0	9.8009	9.8738	9.8426	0.7430	0.0312	0	9.8673	9.9340	9.9010	0.6763	0.0330	0	9.8097	9.8734	9.8187	0.6491	0.0547	0	9.8760	9.9514	9.8751	0.7631	0.0763	0
54	9.8750	9.9325	9.8884	0.5827	0.0442	0	9.7587	9.8380	9.8147	0.8125	0.0233	0	9.8346	9.9071	9.8723	0.7366	0.0348	0	9.7814	9.8500	9.7931	0.7013	0.0570	0	9.8496	9.9321	9.8523	0.8378	0.0798	0
55	9.8407	9.9031	9.8740	0.6335	0.0291	0	9.7120	9.7980	9.7836	0.8860	0.0144	0	9.7990	9.8778	9.8417	0.8043	0.0361	0	9.7516	9.8259	9.7676	0.7614	0.0583	0	9.8196	9.9099	9.8218	0.9199	0.0881	0
56	9.8049	9.8730	9.8293	0.6941	0.0437	0	9.6618	9.7554	9.7344	0.9686	0.0210	0	9.7607	9.8465	9.8096	0.8784	0.0369	0	9.7202	9.8005	9.7375	0.8264	0.0630	0	9.7876	9.8868	9.7985	1.0141	0.0883	0
57	9.7637	9.8381	9.7914	0.7622	0.0468	0	9.6058	9.7073	9.6814	1.0569	0.0259	0	9.7189	9.8119	9.7678	0.9567	0.0440	0	9.6852	9.7720	9.7088	0.8955	0.0631	0	9.7518	9.8606	9.7621	1.1159	0.0985	0
58	9.7226	9.8030	9.7652	0.8272	0.0378	0	9.5458	9.6554	9.6374	1.1482	0.0179	0	9.6731	9.7743	9.7380	1.0466	0.0363	0	9.6479	9.7414	9.6780	0.9693	0.0635	0	9.7135	9.8331	9.7299	1.2313	0.1032	0
59	9.6742	9.7611	9.7131	0.8991	0.0481	0	9.4786	9.5967	9.5830	1.2465	0.0137	0	9.6233	9.7332	9.6961	1.1421	0.0371	0	9.6077	9.7092	9.6536	1.0564	0.0556	0	9.6699	9.8007	9.6928	1.3530	0.1079	0
60	9.6220	9.7167	9.6702	0.9840	0.0465	0	9.4063	9.5336	9.5279	1.3535	0.0058	0	9.5690	9.6887	9.6642	1.2515	0.0246	0	9.5644	9.6747	9.6088	1.1528	0.0659	0	9.6248	9.7684	9.6509	1.4929	0.1175	0
61	9.5638	9.6661	9.6146	1.0694	0.0515	0	9.3300	9.4673	9.4571	1.4718	0.0102	0	9.5104	9.6408	9.6128	1.3716	0.0280	0	9.5184	9.6379	9.5763	1.2549	0.0615	0	9.5714	9.7287	9.6028	1.6438	0.1260	0
62	9.5020	9.6129	9.5690	1.1668	0.0439	0	9.2482	9.3965	9.3933	1.6043	0.0033	0	9.4482	9.5907	9.5561	1.5082	0.0346	0	9.4680	9.5978	9.5219	1.3707	0.0759	0	9.5180	9.6917	9.5457	1.8255	0.1460	0
63	9.4336	9.5534	9.5031	1.2695	0.0503	0	9.1575	9.3165	9.3217	1.7360	-0.0052	1	9.3801	9.5360	9.4888	1.6612	0.0471	0	9.4133	9.5546	9.4744	1.5017	0.0803	0	9.4538	9.6442	9.4750	2.0136	0.1692	0
64	9.3590	9.4881	9.4336	1.3791	0.0545	0	9.0602	9.2308	9.2490	1.8832	-0.0182	1	9.3059	9.4758	9.4250	1.8260	0.0508	0	9.3537	9.5075	9.4181	1.6445	0.0894	0	9.3837	9.5926	9.4182	2.2261	0.1744	0
65	9.2756	9.4143	9.3584	1.4949	0.0559	0	8.9573	9.1412	9.1546	2.0537	-0.0133	1	9.2262	9.4120	9.3367	2.0143	0.0753	0	9.2874	9.4549	9.3506	1.8036	0.1042	0	9.3062	9.5342	9.3565	2.4505	0.1777	0
66	9.1841	9.3333	9.2779	1.6245	0.0554	0	8.8487	9.0459	9.0497	2.2286	-0.0037	1	9.1389	9.3423	9.2515	2.2267	0.0908	0	9.2142	9.3968	9.2797	1.9822	0.1171	0	9.2183	9.4677	9.2639	2.7054	0.2038	0
67	9.0882	9.2489	9.1884	1.7691	0.0605	0	8.7345	8.9472	8.9448	2.4347	0.0024	0	9.0436	9.2659	9.1424	2.4585	0.1235	0	9.1325	9.3314	9.1884	2.1783	0.1431	0	9.1222	9.3960	9.1758	3.0012	0.2202	0
68	8.9832	9.1556	9.0898	1.9193	0.0658	0	8.6116	8.8394	8.8258	2.6452	0.0136	0	8.9399	9.1824	9.0484	2.7127	0.1340	0	9.0418	9.2585	9.1006	2.3962	0.1579	0	9.0146	9.3132	9.0714	3.3128	0.2418	0
69	8.8641	9.0485	8.9722	2.0799	0.0764	0	8.4787	8.7226	8.6810	2.8767	0.0416	0	8.8256	9.0903	8.9299	2.9989	0.1604	0	8.9407	9.1757	9.0024	2.6287	0.1733	0	8.8928	9.2170	8.9503	3.6463	0.2667	0
70	8.7362	8.9323	8.8291	2.2451	0.1032	0	8.3368	8.5967	8.5145	3.1176	0.0821	0	8.7001	8.9878	8.7963	3.3063	0.1914	0	8.8268	9.0818	8.8893	2.8887	0.1925	0	8.7637	9.1153	8.8187	4.0125	0.2966	0
71	8.5963	8.8040	8.6899	2.4167	0.1141	0	8.1884	8.4661	8.3569	3.3923	0.1092	0	8.5618	8.8737	8.6425	3.6432	0.2312	0	8.7001	8.9759	8.7572	3.1695	0.2187	0	8.6098	8.9852	8.6776	4.3602	0.3076	0
72	8.4475	8.6676	8.5228	2.6053	0.1448	0	8.0326	8.3277	8.1704	3.6748	0.1573	0	8.4113	8.7481	8.4846	4.0045	0.2635	0	8.5581	8.8549	8.6228	3.4680	0.2322	0	8.4516	8.8554	8.5102	4.7780	0.3453	0
73	8.2829	8.5132	8.3340	2.7812	0.1793	0	7.8636	8.1756	7.9836	3.9684	0.1920	0	8.2466	8.6080	8.2908	4.3821	0.3172	0	8.4008	8.7191	8.4586	3.7894	0.2605	0	8.2673	8.6934	8.3291	5.1542	0.3643	0
74	8.0997	8.3399	8.1954	2.9652	0.1445	0	7.6844	8.0119	7.7734	4.2625	0.2385	0	8.0664	8.4521	8.1062	4.7815	0.3459	0	8.2264	8.5660	8.2789	4.1279	0.2871	0	8.0710	8.5191	8.1320	5.5522	0.3871	0
75	7.9029	8.1519	7.9943	3.1511	0.1576	0	7.4943	7.8368	7.5462	4.5696	0.2906	0	7.8705	8.2790	7.9058	5.1902	0.3732	0	8.0339	8.3932	8.0958	4.4732	0.2974	0	7.8562	8.3235	7.9207	5.9481	0.4028	0
76	7.6891	7.9447	7.7661	3.3241	0.1787	0	7.2910	7.6472	7.3473	4.8855	0.3000	0	7.6579	8.0878	7.6726	5.6138	0.4152	0	7.8229	8.2009	7.8918	4.8309	0.3091	0	7.6238	8.1060	7.7051	6.3248	0.4009	0
77	7.4608	7.7226	7.5358	3.5088	0.1869	0	7.0754	7.4421	7.1042	5.1841	0.3380	0	7.4309	7.8788	7.4345	6.0273	0.4443	0	7.5934	7.9865	7.6560	5.1774	0.3305	0	7.3776	7.8707	7.4577	6.6844	0.4130	0
78	7.2265	7.4930	7.3000	3.6884	0.1930	0	6.8503	7.2255	6.8808	5.4783	0.3447	0	7.1871	7.6505	7.2051	6.4470	0.4454	0	7.3470	7.7531	7.4276	5.5266	0.3255	0	7.1187	7.6199	7.2083	7.0404	0.4116	0
79	6.9700	7.2365	7.0113	3.8234	0.2251	0	6.6106	6.9914	6.5866	5.7610	0.4048	0	6.9299	7.4031	6.9324	6.8284	0.4707	0	7.0847	7.5000	7.1598	5.8612	0.3402	0	6.8354	7.3293	6.9338	7.2263	0.3956	0
80	6.7045	6.9697	6.7351	3.9549	0.2345	0	6.3590	6.7405	6.3829	6.0001	0.3577	0	6.6613	7.1408	6.6654	7.1986	0.4754	0	6.8066	7.2267	6.8742	6.1709	0.3524	0	6.5568	7.0432	6.6326	7.4181	0.4106	0
81	6.4350	6.6993	6.4079	4.1067	0.2914	0	6.1100	6.4912	5.9044	6.2376	0.5868	0	6.3795	6.8595	6.3386	7.5241	0.5209	0	6.5199	6.9400	6.5666	6.4433	0.3734	0	6.2499	6.7134	6.3381	7.4150	0.3752	0
82	6.1623	6.4234	6.1584	4.2367	0.2650	0	5.8511	6.2273	5.8909	6.4286	0.3363	0	6.0882	6.5623	6.0542	7.7867	0.5081	0	6.2248	6.6395	6.2816	6.6625	0.3579	0	5.9613	6.4093	6.0094	7.5149	0.3998	0
83	5.8795	6.1332	5.8415	4.3148	0.2917	0	5.5861	5.9504	5.5380	6.5218	0.4124	0	5.7890	6.2506	5.7726	7.9736	0.4780	0	5.9221	6.3265	5.9993	6.8289	0.3272	0	5.6502	6.0634	5.7310	7.3116	0.3324	0
84	5.5911	5.8342	5.5351	4.3478	0.2991	0	5.3183	5.6690	5.1813	6.5945	0.4877	0	5.4903	5.9366	5.4799	8.1271	0.4566	0	5.6176	6.0074	5.6693	6.9389	0.3381	0	5.3468	5.7261	5.4276	7.0939	0.2985	0
85	5.2311	5.4449	5.1972	4.0880	0.2477	0	4.9877	5.2966	4.8808	6.1922	0.4158	0	5.1270	5.5224	5.1253	7.7136	0.3971	0	5.2443	5.5907	5.2436	6.6038	0.3471	0	4.9819	5.2998	5.0418	6.3808	0.2579	0
86	4.8814	5.0675	4.7748	3.8135	0.2927	0	4.6654	4.9362	4.5390	5.8031	0.3972	0	4.7734	5.1203	4.7808	7.2694	0.3395	0	4.8803	5.1832	4.9162	6.2063	0.2670	0	4.6311	4.8969	4.7064	5.7384	0.1905	0
87	4.5351	4.6944	4.4311	3.5125	0.2633	0	4.3457	4.5764	4.2364	5.3095	0.3400	0	4.4309	4.7315	4.4134	6.7826	0.3181	0	4.5253	4.7860	4.5486	5.7625	0.2374	0	4.2891	4.5037	4.3602	5.0035	0.1435	0
88	4.1996	4.3347	4.0895	3.2161	0.2451	0	4.0295	4.2245	3.9169	4.8379	0.3076	0	4.0963	4.3517	4.0915	6.2347	0.2603	0	4.1766	4.3970	4.1859	5.2788	0.2111	0	3.9619	4.1314	4.0519	4.2784	0.0796	0
89	3.8596	3.9728	3.8028	2.9334	0.1700	0	3.7141	3.8760	3.6123	4.3606	0.2637	0	3.7666	3.9800	3.7350	5.6667	0.2450	0	3.8370	4.0202	3.8349	4.7760	0.1853	0	3.6360	3.7634	3.7089	3.5032	0.0545	0
90	3.5203	3.6118	3.4631	2.5975	0.1487	0	3.3973	3.5266	3.3223	3.8085	0.2043	0	3.4388	3.6118	3.4317	5.0301	0.1801	0	3.4987	3.6465	3.5028	4.2251	0.1438	0	3.3229	3.4163	3.3755	2.8092	0.0408	0
91	3.1752	3.2484	3.1181	2.3049	0.1302	0	3.0782	3.1793	2.9890	3.2832	0.1903	0	3.1094	3.2448	3.0758	4.3565	0.1690	0	3.1561	3.2714	3.1149	3.6526	0.1565	0	2.9952	3.0546	3.0591	1.9845	-0.0045	1
92	2.8254	2.8839	2.7851	2.0712	0.0988	0	2.7432	2.8192	2.6874	2.7693	0.1318	0	2.7673	2.8686	2.7524	3.6587	0.1162	0	2.8025	2.8875	2.7836	3.0350	0.1039	0	2.6921	2.7438	2.7038	1.9178	0.0400	0
93	2.4497	2.4928	2.4024	1.7617	0.0904	0	2.3819	2.4344	2.3306	2.2035	0.1038	0	2.4019	2.4712	2.3592	2.8844	0.1119	0	2.4240	2.4825	2.3983	2.4117	0.0842	0	2.3396	2.3706	2.3781	1.3278	-0.0075	1
94	2.0179	2.0463	1.9787	1.4047	0.0676	0	1.9733	2.0050	1.9462	1.6078	0.0588	0	1.9883	2.0314	1.9685	2.1721	0.0629	0	1.9987	2.0338	1.9928	1.7571	0.0409	0	1.9440	1.9634	1.9824	1.0020	-0.0190	1
95	1.5045	1.5199	1.4919	1.0252	0.0280	0	1.4815	1.4971	1.4661	1.0538	0.0310	0	1.4898	1.5109	1.4779	1.4130	0.0330	0	1.4939	1.5107	1.4912	1.1213	0.0195	0	1.4679	1.4780	1.4751	0.6841	0.0028	0

LC model: Nonlinear variant

New Zealand

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9282	9.9712	9.9513	0.4326	0.0199	0
51	9.9010	9.9481	9.9191	0.4757	0.0289	0
52	9.8716	9.9232	9.9042	0.5233	0.0190	0
53	9.8399	9.8964	9.8706	0.5743	0.0258	0
54	9.8040	9.8656	9.8562	0.6285	0.0095	0
55	9.7677	9.8352	9.8333	0.6905	0.0019	0
56	9.7260	9.7994	9.7846	0.7546	0.0148	0
57	9.6807	9.7602	9.7223	0.8214	0.0379	0
58	9.6306	9.7177	9.6985	0.9036	0.0191	0
59	9.5731	9.6670	9.6489	0.9807	0.0181	0
60	9.5146	9.6159	9.5913	1.0637	0.0245	0
61	9.4519	9.5613	9.5364	1.1571	0.0248	0
62	9.3846	9.5033	9.4576	1.2647	0.0457	0
63	9.3100	9.4383	9.4041	1.3784	0.0342	0
64	9.2294	9.3676	9.3320	1.4972	0.0355	0
65	9.1422	9.2919	9.2379	1.6369	0.0540	0
66	9.0457	9.2074	9.1430	1.7883	0.0644	0
67	8.9470	9.1218	9.0443	1.9546	0.0775	0
68	8.8409	9.0289	8.9082	2.1262	0.1207	0
69	8.7226	8.9242	8.8277	2.3112	0.0965	0
70	8.5936	8.8096	8.6591	2.5133	0.1505	0
71	8.4518	8.6815	8.5435	2.7179	0.1380	0
72	8.3068	8.5506	8.3845	2.9340	0.1660	0
73	8.1399	8.3958	8.1917	3.1443	0.2042	0
74	7.9592	8.2261	8.0180	3.3532	0.2081	0
75	7.7636	8.0426	7.7942	3.5931	0.2483	0
76	7.5588	7.8491	7.5782	3.8405	0.2710	0
77	7.3310	7.6318	7.3687	4.1027	0.2631	0
78	7.1006	7.4107	7.1080	4.3679	0.3027	0
79	6.8465	7.1619	6.7526	4.6064	0.4092	0
80	6.5811	6.9014	6.5558	4.8674	0.3456	0
81	6.3383	6.6648	6.1917	5.1513	0.4731	0
82	6.0870	6.4170	5.9698	5.4206	0.4472	0
83	5.8332	6.1606	5.6619	5.6131	0.4988	0
84	5.5682	5.8893	5.3041	5.7673	0.5852	0
85	5.2166	5.5069	5.0171	5.5639	0.4898	0
86	4.8830	5.1439	4.5849	5.3430	0.5590	0
87	4.5341	4.7629	4.2225	5.0458	0.5403	0
88	4.2059	4.4069	4.0022	4.7786	0.4047	0
89	3.8716	4.0460	3.6693	4.5050	0.3767	0
90	3.5216	3.6687	3.3595	4.1780	0.3092	0
91	3.1973	3.3191	3.0227	3.8098	0.2965	0
92	2.8625	2.9597	2.6454	3.3943	0.3143	0
93	2.4671	2.5382	2.3352	2.8808	0.2030	0
94	2.0294	2.0756	1.9375	2.2760	0.1381	0
95	1.5065	1.5315	1.4813	1.6543	0.0502	0

Sweden

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9457	9.9689	9.9639	0.2335	0.0050	0
51	9.9225	9.9480	9.9453	0.2562	0.0027	0
52	9.8975	9.9254	9.9227	0.2815	0.0026	0
53	9.8698	9.9004	9.8928	0.3107	0.0076	0
54	9.8390	9.8725	9.8679	0.3408	0.0046	0
55	9.8056	9.8424	9.8361	0.3754	0.0064	0
56	9.7698	9.8105	9.7988	0.4164	0.0117	0
57	9.7295	9.7739	9.7514	0.4565	0.0225	0
58	9.6858	9.7347	9.7172	0.5044	0.0175	0
59	9.6378	9.6921	9.6675	0.5636	0.0246	0
60	9.5847	9.6442	9.6264	0.6200	0.0178	0
61	9.5273	9.5930	9.5637	0.6896	0.0293	0
62	9.4639	9.5363	9.5224	0.7653	0.0139	0
63	9.3956	9.4756	9.4527	0.8516	0.0229	0
64	9.3197	9.4076	9.3851	0.9434	0.0225	0
65	9.2383	9.3347	9.2898	1.0441	0.0450	0
66	9.1495	9.2557	9.1905	1.1604	0.0651	0
67	9.0501	9.1657	9.1074	1.2778	0.0583	0
68	8.9392	9.0656	8.9778	1.4140	0.0878	0
69	8.8164	8.9537	8.8479	1.5578	0.1058	0
70	8.6812	8.8303	8.7466	1.7178	0.0837	0
71	8.5305	8.6911	8.5814	1.8823	0.1097	0
72	8.3697	8.5428	8.4248	2.0676	0.1180	0
73	8.1919	8.3762	8.2555	2.2492	0.1207	0
74	8.0005	8.1973	8.0369	2.4596	0.1604	0
75	7.7961	8.0041	7.8595	2.6675	0.1446	0
76	7.5738	7.7912	7.5841	2.8708	0.2071	0
77	7.3382	7.5655	7.3422	3.0968	0.2233	0
78	7.0898	7.3246	7.0660	3.3127	0.2586	0
79	6.8242	7.0626	6.7528	3.4933	0.3098	0
80	6.5513	6.7939	6.4898	3.7033	0.3041	0
81	6.2674	6.5104	6.1743	3.8770	0.3362	0
82	5.9775	6.2180	5.8939	4.0231	0.3241	0
83	5.6807	5.9162	5.5279	4.1458	0.3883	0
84	5.3827	5.6098	5.2461	4.2204	0.3637	0
85	5.0324	5.2386	4.8590	4.0989	0.3797	0
86	4.6924	4.8764	4.4997	3.9218	0.3767	0
87	4.3572	4.5194	4.1952	3.7210	0.3242	0
88	4.0260	4.1666	3.8725	3.4937	0.2941	0
89	3.7042	3.8255	3.5470	3.2734	0.2785	0
90	3.3823	3.4834	3.2705	2.9890	0.2129	0
91	3.0596	3.1422	2.9281	2.6991	0.2141	0
92	2.7275	2.7922	2.6073	2.3729	0.1849	0
93	2.3698	2.4174	2.2668	2.0125	0.1506	0
94	1.9670	1.9979	1.9003	1.5699	0.0976	0
95	1.4748	1.4909	1.4420	1.0897	0.0489	0

Germany

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.8940	9.9339	9.8861	0.4033	0.0478	0
51	9.8704	9.9143	9.8616	0.4446	0.0527	0
52	9.8447	9.8929	9.8363	0.4896	0.0566	0
53	9.8157	9.8687	9.8068	0.5399	0.0619	0
54	9.7829	9.8411	9.7731	0.5957	0.0680	0
55	9.7456	9.8090	9.7488	0.6508	0.0603	0
56	9.7026	9.7713	9.6991	0.7080	0.0721	0
57	9.6535	9.7275	9.6588	0.7666	0.0686	0
58	9.5987	9.6782	9.6370	0.8273	0.0412	0
59	9.5374	9.6229	9.5802	0.8964	0.0427	0
60	9.4705	9.5617	9.5597	0.9625	0.0020	0
61	9.3997	9.4971	9.4789	1.0370	0.0182	0
62	9.3241	9.4284	9.4277	1.1181	0.0007	0
63	9.2437	9.3559	9.3652	1.2135	-0.0093	1
64	9.1600	9.2803	9.2773	1.3135	0.0030	0
65	9.0709	9.2011	9.1905	1.4354	0.0105	0
66	8.9790	9.1196	9.0496	1.5661	0.0700	0
67	8.8787	9.0318	8.9766	1.7244	0.0553	0
68	8.7735	8.9397	8.8504	1.8940	0.0893	0
69	8.6608	8.8422	8.7027	2.0945	0.1395	0
70	8.5348	8.7323	8.5501	2.3145	0.1822	0
71	8.3945	8.6079	8.4011	2.5422	0.2068	0
72	8.2394	8.4703	8.2249	2.8025	0.2454	0
73	8.0674	8.3147	8.0629	3.0654	0.2518	0
74	7.8735	8.1377	7.8924	3.3549	0.2452	0
75	7.6649	7.9467	7.6943	3.6757	0.2524	0
76	7.4392	7.7378	7.4687	4.0140	0.2691	0
77	7.1944	7.5060	7.2441	4.3304	0.2619	0
78	6.9368	7.2610	6.9665	4.6747	0.2946	0
79	6.6628	6.9978	6.6449	5.0275	0.3529	0
80	6.3753	6.7165	6.3747	5.3514	0.3418	0
81	6.0773	6.4213	5.8474	5.6603	0.5739	0
82	5.7637	6.1043	6.0839	5.9108	0.0204	0
83	5.4568	5.7944	5.5173	6.1867	0.2771	0
84	5.1550	5.4850	5.1217	6.4019	0.3633	0
85	4.8043	5.1038	4.9300	6.2330	0.1737	0
86	4.4736	4.7433	4.3553	6.0282	0.3880	0
87	4.1427	4.3825	4.1629	5.7886	0.2197	0
88	3.8271	4.0385	3.8505	5.5257	0.1880	0
89	3.5178	3.7026	3.5331	5.2542	0.1695	0
90	3.2190	3.3783	3.2417	4.9491	0.1366	0
91	2.9176	3.0489	2.9286	4.4998	0.1203	0
92	2.6085	2.7129	2.6256	4.0046	0.0873	0
93	2.2800	2.3589	2.2867	3.4585	0.0721	0
94	1.8997	1.9525	1.9028	2.7775	0.0496	0
95	1.4411	1.4691	1.4466	1.9399	0.0225	0

Russia

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.1434	9.4160	9.2446	2.9823	0.1714	0
51	9.0988	9.3733	9.1729	3.0176	0.2004	0
52	9.0496	9.3241	9.0677	3.0336	0.2564	0
53	8.9900	9.2688	9.1476	3.1010	0.1212	0
54	8.9300	9.2094	8.9133	3.1291	0.2961	0
55	8.8655	9.1459	9.0015	3.1629	0.1444	0
56	8.8020	9.0804	8.8733	3.1625	0.2071	0
57	8.7306	9.0086	8.8001	3.1843	0.2085	0
58	8.6544	8.9302	8.8848	3.1862	0.0454	0
59	8.5737	8.8458	8.5613	3.1748	0.2845	0
60	8.4889	8.7555	8.6315	3.1409	0.1240	0
61	8.4000	8.6617	8.5191	3.1158	0.1426	0
62	8.3152	8.5639	8.3686	2.9916	0.1953	0
63	8.2126	8.4533	8.3787	2.9308	0.0746	0
64	8.1066	8.3368	8.1978	2.8399	0.1390	0
65	7.9931	8.2119	8.0594	2.7367	0.1525	0
66	7.8689	8.0770	7.9169	2.6445	0.1601	0
67	7.7419	7.9387	7.8550	2.5420	0.0837	0
68	7.5941	7.7862	7.6654	2.5299	0.1208	0
69	7.4373	7.6262	7.4584	2.5405	0.1678	0
70	7.2749	7.4622	7.3445	2.5741	0.1177	0
71	7.0833	7.2753	7.1416	2.7111	0.1337	0
72	6.9044	7.0957	6.9569	2.7708	0.1388	0
73	6.6876	6.8899	6.7232	3.0253	0.1668	0
74	6.4579	6.6705	6.5745	3.2922	0.0960	0
75	6.2237	6.4464	6.3831	3.5791	0.0633	0
76	5.9735	6.2054	6.1887	3.8824	0.0167	0
77	5.7207	5.9625	5.8660	4.2258	0.0964	0
78	5.4677	5.7150	5.8095	4.5228	-0.0945	1
79	5.1893	5.4458	5.4582	4.9417	-0.0125	1
80	4.9392	5.2039	5.1802	5.3586	0.0237	0
81	4.6641	4.9406	4.8733	5.9290	0.0673	0
82	4.4368	4.7208	4.6195	6.4016	0.1014	0
83	4.1815	4.4872	4.5215	7.3121	-0.0343	1
84	3.9434	4.2692	4.2097	8.2616	0.0595	0
85	3.7051	4.0339	3.9581	8.8745	0.0758	0
86	3.4674	3.7952	3.6299	9.4525	0.1653	0
87	3.2443	3.5721	3.4169	10.1033	0.1552	0
88	3.0258	3.3487	3.2340	10.6732	0.1147	0
89	2.8122	3.1260	2.9343	11.1593	0.1917	0
90	2.6277	2.9168	2.7585	11.0033	0.1582	0
91	2.3899	2.6691	2.5201	11.6785	0.1490	0
92	2.2287	2.4387	2.2914	9.4233	0.1473	0
93	1.9991	2.1659	2.0314	8.3450	0.1345	0
94	1.7251	1.8376	1.7411	6.5192	0.0965	0
95	1.3638	1.4178	1.3702	3.9573	0.0477	0

CBD model: Linear variant

x	Australia $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	United Kingdom $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	Italy $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	France $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	Spain $L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9907	10.0247	9.9828	0.3402	0.0418	0	9.9170	9.9546	9.9201	0.3790	0.0345	0	9.9721	10.0135	9.9729	0.4157	0.0406	0	9.9276	9.9914	9.8781	0.6425	0.1133	0	9.9727	10.0315	9.9400	0.5895	0.0915	0
51	9.9686	10.0064	9.9623	0.3792	0.0441	0	9.8884	9.9300	9.8976	0.4206	0.0324	0	9.9483	9.9950	9.9500	0.4689	0.0449	0	9.9018	9.9717	9.8599	0.7053	0.1117	0	9.9489	10.0146	9.9207	0.6602	0.0939	0
52	9.9440	9.9862	9.9488	0.4237	0.0373	0	9.8567	9.9032	9.8747	0.4712	0.0285	0	9.9221	9.9743	9.9312	0.5264	0.0431	0	9.8735	9.9501	9.8402	0.7755	0.1099	0	9.9225	9.9951	9.8990	0.7317	0.0961	0
53	9.9168	9.9636	9.9250	0.4722	0.0386	0	9.8219	9.8738	9.8426	0.5282	0.0312	0	9.8930	9.9513	9.9010	0.5891	0.0502	0	9.8421	9.9261	9.8187	0.8527	0.1074	0	9.8935	9.9741	9.8751	0.8150	0.0990	0
54	9.8865	9.9389	9.8884	0.5302	0.0505	0	9.7834	9.8410	9.8147	0.5888	0.0263	0	9.8608	9.9260	9.8723	0.6617	0.0538	0	9.8081	9.9003	9.7931	0.9399	0.1072	0	9.8612	9.9511	9.8523	0.9123	0.0988	0
55	9.8530	9.9117	9.8740	0.5957	0.0377	0	9.7411	9.8053	9.7836	0.6586	0.0216	0	9.8251	9.8980	9.8417	0.7419	0.0564	0	9.7705	9.8710	9.7676	1.0289	0.1034	0	9.8253	9.9251	9.8218	1.0156	0.1033	0
56	9.8157	9.8811	9.8293	0.6657	0.0518	0	9.6945	9.7662	9.7344	0.7394	0.0318	0	9.7857	9.8669	9.8096	0.8295	0.0573	0	9.7295	9.8401	9.7375	1.1364	0.1025	0	9.7857	9.8959	9.7985	1.1260	0.0974	0
57	9.7746	9.8471	9.7914	0.7427	0.0558	0	9.6433	9.7236	9.6814	0.8334	0.0422	0	9.7420	9.8333	9.7678	0.9374	0.0655	0	9.6846	9.8058	9.7088	1.2521	0.0970	0	9.7420	9.8652	9.7621	1.2647	0.1031	0
58	9.7290	9.8105	9.7652	0.8382	0.0453	0	9.5869	9.6765	9.6374	0.9344	0.0391	0	9.6940	9.7959	9.7380	1.0510	0.0579	0	9.6353	9.7673	9.6780	1.3707	0.0894	0	9.6936	9.8294	9.7299	1.4001	0.0995	0
59	9.6786	9.7693	9.7131	0.9365	0.0562	0	9.5252	9.6252	9.5830	1.0491	0.0422	0	9.6409	9.7542	9.6961	1.1747	0.0581	0	9.5817	9.7266	9.6536	1.5128	0.0730	0	9.6405	9.7915	9.6928	1.5669	0.0988	0
60	9.6226	9.7243	9.6702	1.0570	0.0542	0	9.4574	9.5683	9.5279	1.1728	0.0405	0	9.5824	9.7091	9.6642	1.3217	0.0449	0	9.5235	9.6816	9.6088	1.6597	0.0728	0	9.5814	9.7475	9.6509	1.7333	0.0966	0
61	9.5612	9.6744	9.6146	1.1839	0.0598	0	9.3833	9.5074	9.4571	1.3227	0.0503	0	9.5182	9.6592	9.6128	1.4816	0.0465	0	9.4595	9.6325	9.5763	1.8289	0.0561	0	9.5168	9.7003	9.6028	1.9286	0.0975	0
62	9.4935	9.6202	9.5690	1.3346	0.0512	0	9.3022	9.4398	9.3933	1.4794	0.0466	0	9.4473	9.6042	9.5561	1.6607	0.0481	0	9.3898	9.5771	9.5219	1.9952	0.0552	0	9.4456	9.6486	9.5457	2.1489	0.1029	0
63	9.4188	9.5604	9.5031	1.5038	0.0573	0	9.2139	9.3674	9.3217	1.6661	0.0457	0	9.3697	9.5446	9.4888	1.8668	0.0558	0	9.3141	9.5195	9.4744	2.2047	0.0451	0	9.3673	9.5914	9.4750	2.3920	0.1164	0
64	9.3367	9.4953	9.4336	1.6988	0.0618	0	9.1174	9.2869	9.2490	1.8584	0.0378	0	9.2846	9.4788	9.4250	2.0909	0.0537	0	9.2319	9.4544	9.4181	2.4099	0.0363	0	9.2814	9.5284	9.4182	2.6612	0.1102	0
65	9.2468	9.4221	9.3584	1.8968	0.0638	0	9.0126	9.2007	9.1546	2.0876	0.0462	0	9.1918	9.4078	9.3367	2.3504	0.0711	0	9.1422	9.3838	9.3506	2.6423	0.0332	0	9.1879	9.4596	9.3565	2.9574	0.1031	0
66	9.1482	9.3451	9.2779	2.1521	0.0672	0	8.8989	9.1072	9.0497	2.3416	0.0575	0	9.0899	9.3281	9.2515	2.6197	0.0765	0	9.0459	9.3086	9.2797	2.9049	0.0289	0	9.0856	9.3852	9.2639	3.2976	0.1213	0
67	9.0406	9.2584	9.1884	2.4093	0.0700	0	8.7759	9.0061	8.9448	2.6238	0.0614	0	8.9790	9.2440	9.1424	2.9516	0.1015	0	8.9415	9.2254	9.1884	3.1745	0.0370	0	8.9739	9.3009	9.1758	3.6437	0.1251	0
68	8.9230	9.1642	9.0898	2.7032	0.0744	0	8.6428	8.8950	8.8258	2.9180	0.0692	0	8.8587	9.1511	9.0484	3.3003	0.1027	0	8.8288	9.1357	9.1006	3.4750	0.0350	0	8.8524	9.2087	9.0714	4.0249	0.1373	0
69	8.7954	9.0611	8.9722	3.0217	0.0890	0	8.4997	8.7766	8.6810	3.2574	0.0956	0	8.7279	9.0490	8.9299	3.6793	0.1192	0	8.7075	9.0382	9.0024	3.7982	0.0358	0	8.7210	9.1107	8.9503	4.4678	0.1603	0
70	8.6571	8.9511	8.8291	3.3968	0.1220	0	8.3458	8.6496	8.5145	3.6394	0.1350	0	8.5870	8.9383	8.7963	4.0913	0.1419	0	8.5768	8.9339	8.8893	4.1628	0.0446	0	8.5783	9.0021	8.8187	4.9406	0.1834	0
71	8.5073	8.8311	8.6899	3.8067	0.1412	0	8.1815	8.5122	8.3569	4.0424	0.1552	0	8.4346	8.8206	8.6425	4.5756	0.1781	0	8.4379	8.8207	8.7572	4.5369	0.0635	0	8.4248	8.8859	8.6776	5.4728	0.2083	0
72	8.3458	8.7013	8.5228	4.2598	0.1786	0	8.0059	8.3658	8.1704	4.4944	0.1954	0	8.2710	8.6933	8.4846	5.1053	0.2087	0	8.2885	8.6973	8.6228	4.9318	0.0745	0	8.2595	8.7548	8.5102	5.9963	0.2446	0
73	8.1725	8.5609	8.3340	4.7522	0.2270	0	7.8192	8.2086	7.9836	4.9793	0.2249	0	8.0956	8.5513	8.2908	5.6287	0.2605	0	8.1295	8.5665	8.4586	5.3755	0.1079	0	8.0842	8.6213	8.3291	6.6431	0.2922	0
74	7.9869	8.4118	8.1954	5.3209	0.2164	0	7.6219	8.0415	7.7734	5.5054	0.2680	0	7.9086	8.4060	8.1062	6.2893	0.2998	0	7.9613	8.4277	8.2789	5.8574	0.1488	0	7.8954	8.4727	8.1320	7.3129	0.3408	0
75	7.7893	8.2499	7.9943	5.9128	0.2556	0	7.4134	7.8629	7.5462	6.0625	0.3166	0	7.7103	8.2428	7.9058	6.9061	0.3370	0	7.7821	8.2768	8.0958	6.3573	0.1810	0	7.6955	8.3102	7.9207	7.9877	0.3895	0
76	7.5803	8.0745	7.7661	6.5196	0.3084	0	7.1953	7.6759	7.3473	6.6790	0.3286	0	7.5002	8.0740	7.6726	7.6499	0.4014	0	7.5937	8.1134	7.8918	6.8439	0.2216	0	7.4838	8.1400	7.7051	8.7690	0.4349	0
77	7.3591	7.8918	7.5358	7.2391	0.3561	0	6.9677	7.4787	7.1042	7.3348	0.3746	0	7.2795	7.8926	7.4345	8.4233	0.4581	0	7.3949	7.9459	7.6560	7.4514	0.2899	0	7.2621	7.9569	7.4577	9.5680	0.4992	0
78	7.1270	7.6971	7.3000	8.0002	0.3971	0	6.7313	7.2710	6.8808	8.0182	0.3902	0	7.0477	7.6969	7.2051	9.2114	0.4918	0	7.1884	7.7688	7.4276	8.0734	0.3412	0	7.0294	7.7626	7.2083	10.4311	0.5544	0
79	6.8849	7.4871	7.0113	8.7471	0.4757	0	6.4873	7.0524	6.5866	8.7113	0.4659	0	6.8069	7.4957	6.9324	10.1202	0.5633	0	6.9711	7.5761	7.1598	8.6798	0.4164	0	6.7880	7.5577	6.9338	11.3379	0.6239	0
80	6.6342	7.2734	6.7351	9.6347	0.5383	0	6.2363	6.8248	6.3829	9.4370	0.4420	0	6.5575	7.2781	6.6654	10.9903	0.6127	0	6.7473	7.3734	6.8742	9.2802	0.4992	0	6.5372	7.3416	6.6326	12.3050	0.7090	0
81	6.3746	7.0403	6.4079	10.4438	0.6324	0	5.9804	6.5901	5.9044	10.1959	0.6857	0	6.3010	7.0525	6.3386	11.9260	0.7139	0	6.5167	7.1667	6.5666	9.9748	0.6001	0	6.2807	7.1094	6.3381	13.1948	0.7712	0
82	6.1094	6.8056	6.1584	11.3947	0.6472	0	5.7206	6.3492	5.8909	10.9870	0.4582	0	6.0382	6.8216	6.0542	12.9736	0.7674	0	6.2799	6.9497	6.2816	10.6660	0.6681	0	6.0157	6.8769	6.0094	14.3174	0.8675	0
83	5.8395	6.5596	5.8415	12.3320	0.7181	0	5.4598	6.1006	5.5380	11.7371	0.5626	0	5.7722	6.5819	5.7726	14.0279	0.8093	0	6.0372	6.7243	5.9993	11.3804	0.7249	0	5.7502	6.6259	5.7310	15.2296	0.8949	0
84	5.5667	6.3067	5.5351	13.2928	0.7716	0	5.1979	5.8470	5.1813	12.4877	0.6657	0	5.5028	6.3261	5.4799	14.9612	0.8462	0	5.7913	6.4946	5.6693	12.1448	0.8253	0	5.4806	6.3691	5.4276	16.2118	0.9414	0
85	5.2206	5.9016	5.1972	13.0451	0.7044	0	4.8810	5.4805	4.8808	12.2806	0.5997	0	5.1628	5.9182	5.1253	14.6324	0.7929	0	5.4476	6.1000	5.2436	11.9770	0.8564	0	5.1442	5.9614	5.0418	15.8866	0.9196	0
86	4.8812	5.5018	4.7748	12.7148	0.7270	0	4.5715	5.1199	4.5390	11.9968	0.5809	0	4.8295	5.5207	4.7808	14.3115	0.7399	0	5.1077	5.7129	4.9162	11.8469	0.7967	0	4.8145	5.5694	4.7064	15.6790	0.8631	0
87	4.5482	5.1088	4.4311	12.3264	0.6778	0	4.2675	4.7586	4.2364	11.5070	0.5222	0	4.5014	5.1271	4.4134	13.9011	0.7137	0	4.7681	5.3238	4.5486	11.6542	0.7752	0	4.4879	5.1672	4.3602	15.1366	0.8070	0
88	4.2172	4.7179	4.0895	11.8734	0.6284	0	3.9671	4.4062	3.9169	11.0692	0.4893	0	4.1759	4.7273	4.0915	13.2045	0.6359	0	4.4301	4.9274	4.1859	11.2265	0.7415	0	4.1657	4.7746	4.0519	14.6160	0.7227	0
89	3.8874	4.3262	3.8028	11.2884	0.5234	0	3.6683	4.0488	3.6123	10.3719	0.4365	0	3.8521	4.3321	3.7350	12.4597	0.5971	0	4.0892	4.5275	3.8349	10.7195	0.6926	0	3.8445	4.3808	3.7089	13.9495	0.6719	0
90	3.5548	3.9254	3.4631	10.4256	0.4623	0	3.3667	3.6928	3.3223	9.6856	0.3705	0	3.5246	3.9304	3.4317	11.5146	0.4987	0	3.7412	4.1222	3.5028	10.1838	0.6195	0	3.5195	3.9779	3.3755	13.0225	0.6024	0
91	3.2139	3.5156	3.1181	9.3887	0.3974	0	3.0574	3.3237	2.9890	8.7105	0.3347	0	3.1890	3.5244	3.0758	10.5174	0.4485	0	3.3801	3.7026	3.1149	9.5406	0.5877	0	3.1864	3.5645	3.0591	11.8653	0.5054	0
92	2.8553	3.0953	2.7851	8.4087	0.3103	0	2.7311	2.9398	2.6874	7.6410	0.2524	0	2.8361	3.0961	2.7524	9.1661	0.3436	0	2.9966	3.2522	2.7836	8.5298	0.4686	0	2.8342	3.1347	2.7038	10.6038	0.4310	0
93	2.4670	2.6407	2.4024	7.0414	0.2383	0	2.3745	2.5246	2.3306	6.3190	0.1939	0	2.4525	2.6421	2.3592	7.7315	0.2828	0	2.5792	2.7663	2.3983	7.2529	0.3680	0	2.4528	2.6698	2.3781	8.8474	0.2917	0
94	2.0294	2.1421	1.9787	5.5506	0.1634	0	1.9678	2.0642	1.9462	4.8973	0.1180	0	2.0198	2.1420	1.9685	6.0533	0.1735	0	2.1079	2.2322	1.9928	5.8956	0.2394	0	2.0204	2.1631	1.9824	7.0664	0.1807	0
95	1.5106	1.5686	1.4919	3.8397	0.0767	0	1.4781	1.5273	1.4661	3.3245	0.0612	0	1.5058	1.5671	1.4779	4.0734	0.0893	0	1.5549	1.6209	1.4912	4.2410	0.1297	0	1.5062	1.5813	1.4751	4.9850	0.1061	0

CBD model: Linear variant

	New Zealand						Sweden						Germany						Russia					
x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9407	9.9946	9.9513	0.5427	0.0434	0	9.9759	10.0079	9.9639	0.3210	0.0440	0	9.9222	9.9794	9.8861	0.5769	0.0934	0	9.2802	9.5875	9.2446	3.3114	0.3429	0
51	9.9148	9.9735	9.9191	0.5922	0.0544	0	9.9519	9.9874	9.9453	0.3562	0.0421	0	9.8945	9.9575	9.8616	0.6369	0.0960	0	9.2122	9.5352	9.1729	3.5063	0.3622	0
52	9.8862	9.9504	9.9042	0.6494	0.0462	0	9.9255	9.9647	9.9227	0.3942	0.0419	0	9.8637	9.9324	9.8363	0.6967	0.0962	0	9.1381	9.4813	9.0677	3.7550	0.4136	0
53	9.8549	9.9254	9.8706	0.7155	0.0548	0	9.8962	9.9398	9.8928	0.4406	0.0470	0	9.8299	9.9054	9.8068	0.7686	0.0986	0	9.0600	9.4213	9.1476	3.9876	0.2737	0
54	9.8203	9.8980	9.8562	0.7903	0.0418	0	9.8635	9.9113	9.8679	0.4841	0.0433	0	9.7928	9.8755	9.7731	0.8442	0.1024	0	8.9764	9.3607	8.9133	4.2808	0.4474	0
55	9.7822	9.8675	9.8333	0.8718	0.0342	0	9.8274	9.8805	9.8361	0.5396	0.0444	0	9.7518	9.8426	9.7488	0.9310	0.0938	0	8.8880	9.2951	9.0015	4.5811	0.2937	0
56	9.7403	9.8335	9.7846	0.9570	0.0489	0	9.7873	9.8461	9.7988	0.6005	0.0473	0	9.7067	9.8057	9.6991	1.0198	0.1066	0	8.7935	9.2197	8.8733	4.8460	0.3463	0
57	9.6944	9.7972	9.7223	1.0607	0.0749	0	9.7431	9.8078	9.7514	0.6639	0.0564	0	9.6573	9.7656	9.6588	1.1218	0.1068	0	8.6941	9.1440	8.8001	5.1752	0.3439	0
58	9.6438	9.7572	9.6985	1.1760	0.0586	0	9.6939	9.7660	9.7172	0.7441	0.0488	0	9.6027	9.7209	9.6370	1.2312	0.0839	0	8.5877	9.0638	8.8848	5.5446	0.1790	0
59	9.5880	9.7122	9.6489	1.2952	0.0633	0	9.6396	9.7192	9.6675	0.8261	0.0517	0	9.5430	9.6721	9.5802	1.3523	0.0919	0	8.4753	8.9696	8.5613	5.8322	0.4082	0
60	9.5271	9.6642	9.5913	1.4383	0.0728	0	9.5797	9.6676	9.6264	0.9176	0.0413	0	9.4778	9.6194	9.5597	1.4936	0.0597	0	8.3572	8.8766	8.6315	6.2147	0.2451	0
61	9.4607	9.6114	9.5364	1.5925	0.0750	0	9.5135	9.6113	9.5637	1.0279	0.0477	0	9.4063	9.5604	9.4789	1.6374	0.0815	0	8.2314	8.7720	8.5191	6.5682	0.2529	0
62	9.3877	9.5539	9.4576	1.7699	0.0963	0	9.4408	9.5490	9.5224	1.1464	0.0266	0	9.3280	9.4974	9.4277	1.8158	0.0697	0	8.1003	8.6669	8.3686	6.9943	0.2982	0
63	9.3079	9.4905	9.4041	1.9616	0.0864	0	9.3608	9.4802	9.4527	1.2764	0.0275	0	9.2429	9.4263	9.3652	1.9832	0.0611	0	7.9625	8.5585	8.3787	7.4855	0.1798	0
64	9.2210	9.4222	9.3320	2.1819	0.0902	0	9.2730	9.4054	9.3851	1.4281	0.0204	0	9.1497	9.3495	9.2773	2.1844	0.0722	0	7.8177	8.4355	8.1978	7.9024	0.2377	0
65	9.1262	9.3465	9.2379	2.4144	0.1086	0	9.1766	9.3217	9.2898	1.5814	0.0320	0	9.0487	9.2643	9.1905	2.3821	0.0737	0	7.6667	8.3072	8.0594	8.3548	0.2478	0
66	9.0234	9.2652	9.1430	2.6801	0.1222	0	9.0711	9.2317	9.1905	1.7701	0.0411	0	8.9396	9.1756	9.0496	2.6394	0.1260	0	7.5091	8.1717	7.9169	8.8245	0.2548	0
67	8.9122	9.1770	9.0443	2.9704	0.1326	0	8.9566	9.1336	9.1074	1.9765	0.0262	0	8.8207	9.0759	8.9766	2.8940	0.0993	0	7.3450	8.0357	7.8550	9.4038	0.1807	0
68	8.7914	9.0828	8.9082	3.3153	0.1746	0	8.8310	9.0255	8.9778	2.2020	0.0476	0	8.6918	8.9664	8.8504	3.1587	0.1160	0	7.1752	7.8856	7.6654	9.9007	0.2202	0
69	8.6615	8.9802	8.8277	3.6792	0.1525	0	8.6955	8.9078	8.8479	2.4410	0.0598	0	8.5546	8.8534	8.7027	3.4934	0.1507	0	6.9988	7.7341	7.4584	10.5064	0.2757	0
70	8.5215	8.8681	8.6591	4.0674	0.2090	0	8.5484	8.7811	8.7466	2.7218	0.0345	0	8.4053	8.7271	8.5501	3.8276	0.1770	0	6.8190	7.5691	7.3445	11.0009	0.2246	0
71	8.3709	8.7492	8.5435	4.5197	0.2057	0	8.3899	8.6435	8.5814	3.0230	0.0621	0	8.2464	8.5919	8.4011	4.1901	0.1907	0	6.6323	7.4013	7.1416	11.5943	0.2597	0
72	8.2094	8.6210	8.3845	5.0136	0.2365	0	8.2188	8.4936	8.4248	3.3429	0.0688	0	8.0772	8.4472	8.2249	4.5817	0.2223	0	6.4424	7.2303	6.9569	12.2304	0.2734	0
73	8.0382	8.4851	8.1917	5.5597	0.2934	0	8.0364	8.3350	8.2555	3.7156	0.0795	0	7.8964	8.2910	8.0629	4.9972	0.2281	0	6.2471	7.0462	6.7232	12.7914	0.3231	0
74	7.8561	8.3379	8.0180	6.1333	0.3199	0	7.8416	8.1630	8.0369	4.0981	0.1261	0	7.7040	8.1249	7.8924	5.4633	0.2325	0	6.0514	6.8648	6.5745	13.4421	0.2903	0
75	7.6631	8.1796	7.7942	6.7403	0.3854	0	7.6347	7.9822	7.8595	4.5520	0.1227	0	7.5017	7.9482	7.6943	5.9514	0.2539	0	5.8496	6.6762	6.3831	14.1303	0.2931	0
76	7.4582	8.0101	7.5782	7.4000	0.4319	0	7.4156	7.7874	7.5841	5.0147	0.2034	0	7.2878	7.7609	7.4687	6.4914	0.2921	0	5.6487	6.4808	6.1887	14.7302	0.2921	0
77	7.2448	7.8348	7.3687	8.1435	0.4661	0	7.1858	7.5809	7.3422	5.4991	0.2387	0	7.0656	7.5629	7.2441	7.0385	0.3188	0	5.4459	6.2826	5.8660	15.3644	0.4166	0
78	7.0218	7.6470	7.1080	8.9032	0.5390	0	6.9446	7.3619	7.0660	6.0099	0.2959	0	6.8348	7.3558	6.9665	7.6219	0.3893	0	5.2419	6.0816	5.8095	16.0179	0.2720	0
79	6.7894	7.4508	6.7526	9.7408	0.6981	0	6.6944	7.1329	6.7528	6.5503	0.3800	0	6.5947	7.1387	6.6449	8.2491	0.4937	0	5.0402	5.8771	5.4582	16.6051	0.4189	0
80	6.5491	7.2490	6.5558	10.6856	0.6932	0	6.4359	6.8963	6.4898	7.1540	0.4065	0	6.3471	6.9093	6.3747	8.8572	0.5346	0	4.8390	5.6674	5.1802	17.1195	0.4872	0
81	6.3038	7.0357	6.1917	11.6108	0.8440	0	6.1709	6.6511	6.1743	7.7813	0.4768	0	6.0939	6.6722	5.8474	9.4888	0.8248	0	4.6386	5.4685	4.8733	17.8919	0.5952	0
82	6.0510	6.8083	5.9698	12.5168	0.8386	0	5.9001	6.3914	5.8939	8.3259	0.4975	0	5.8370	6.4297	6.0839	10.1537	0.3458	0	4.4401	5.2579	4.6195	18.4163	0.6384	0
83	5.7946	6.5797	5.6619	13.5492	0.9178	0	5.6249	6.1251	5.5279	8.8918	0.5972	0	5.5770	6.1785	5.5173	10.7850	0.6612	0	4.2482	5.0565	4.5215	19.0283	0.5350	0
84	5.5373	6.3449	5.3041	14.5838	1.0408	0	5.3498	5.8634	5.2461	9.6010	0.6173	0	5.3141	5.9260	5.1217	11.5152	0.8043	0	4.0587	4.8469	4.2097	19.4204	0.6372	0
85	5.2023	5.9518	5.0171	14.4073	0.9347	0	5.0128	5.4896	4.8590	9.5098	0.6306	0	4.9880	5.5576	4.9300	11.4208	0.6276	0	3.8539	4.6019	3.9581	19.4081	0.6438	0
86	4.8746	5.5649	4.5849	14.1619	0.9801	0	4.6837	5.1221	4.4997	9.3584	0.6223	0	4.6686	5.2025	4.3553	11.4365	0.8472	0	3.6536	4.3451	3.6299	18.9265	0.7152	0
87	4.5498	5.1772	4.2225	13.7880	0.9546	0	4.3615	4.7598	4.1952	9.1324	0.5646	0	4.3538	4.8370	4.1629	11.0975	0.6741	0	3.4569	4.0936	3.4169	18.4203	0.6767	0
88	4.2279	4.7899	4.0022	13.2927	0.7877	0	4.0435	4.4045	3.8725	8.9272	0.5319	0	4.0438	4.4863	3.8505	10.9404	0.6357	0	3.2612	3.8441	3.2340	17.8745	0.6101	0
89	3.9043	4.3978	3.6693	12.6390	0.7285	0	3.7284	4.0455	3.5470	8.5036	0.4985	0	3.7347	4.1269	3.5331	10.5001	0.5937	0	3.0647	3.5882	2.9343	17.0810	0.6539	0
90	3.5763	4.0001	3.3595	11.8504	0.6406	0	3.4117	3.6904	3.2705	8.1692	0.4199	0	3.4240	3.7672	3.2417	10.0223	0.5255	0	2.8622	3.3274	2.7585	16.2515	0.5688	0
91	3.2384	3.5882	3.0227	10.8001	0.5655	0	3.0888	3.3211	2.9281	7.5195	0.3930	0	3.1039	3.3921	2.9286	9.2849	0.4636	0	2.6482	3.0468	2.5201	15.0527	0.5267	0
92	2.8788	3.1540	2.6454	9.5591	0.5086	0	2.7521	2.9393	2.6073	6.8033	0.3320	0	2.7673	3.0015	2.6256	8.4635	0.3759	0	2.4149	2.7423	2.2914	13.5573	0.4509	0
93	2.4888	2.6909	2.3352	8.1172	0.3557	0	2.3859	2.5266	2.2668	5.8971	0.2598	0	2.4026	2.5811	2.2867	7.4293	0.2943	0	2.1492	2.3963	2.0314	11.4970	0.3648	0
94	2.0462	2.1797	1.9375	6.5244	0.2422	0	1.9728	2.0658	1.9003	4.7145	0.1654	0	1.9854	2.1064	1.9028	6.0951	0.2036	0	1.8233	1.9916	1.7411	9.2355	0.2505	0
95	1.5211	1.5893	1.4813	4.4807	0.1080	0	1.4793	1.5282	1.4420	3.3048	0.0862	0	1.4872	1.5524	1.4466	4.3831	0.1058	0	1.4051	1.4993	1.3702	6.7047	0.1292	0

CBD model: Nonlinear variant

Australia

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	10.0059	10.0317	9.9828	0.2578	0.0489	0
51	9.9847	10.0134	9.9623	0.2870	0.0510	0
52	9.9610	9.9929	9.9488	0.3203	0.0441	0
53	9.9346	9.9702	9.9250	0.3580	0.0451	0
54	9.9051	9.9450	9.8884	0.4025	0.0566	0
55	9.8723	9.9169	9.8740	0.4520	0.0430	0
56	9.8357	9.8859	9.8293	0.5100	0.0566	0
57	9.7950	9.8515	9.7914	0.5764	0.0601	0
58	9.7498	9.8134	9.7652	0.6523	0.0482	0
59	9.6996	9.7713	9.7131	0.7394	0.0582	0
60	9.6438	9.7249	9.6702	0.8402	0.0547	0
61	9.5821	9.6737	9.6146	0.9561	0.0591	0
62	9.5137	9.6173	9.5690	1.0894	0.0483	0
63	9.4381	9.5550	9.5031	1.2384	0.0518	0
64	9.3547	9.4867	9.4336	1.4115	0.0532	0
65	9.2628	9.4121	9.3584	1.6112	0.0537	0
66	9.1618	9.3301	9.2779	1.8369	0.0522	0
67	9.0510	9.2404	9.1884	2.0921	0.0520	0
68	8.9298	9.1428	9.0898	2.3850	0.0530	0
69	8.7976	9.0356	8.9722	2.7057	0.0634	0
70	8.6536	8.9195	8.8291	3.0722	0.0904	0
71	8.4976	8.7936	8.6899	3.4841	0.1037	0
72	8.3289	8.6569	8.5228	3.9377	0.1341	0
73	8.1474	8.5098	8.3340	4.4472	0.1758	0
74	7.9529	8.3515	8.1954	5.0116	0.1560	0
75	7.7454	8.1817	7.9943	5.6332	0.1874	0
76	7.5251	7.9994	7.7661	6.3027	0.2333	0
77	7.2926	7.8053	7.5358	7.0312	0.2696	0
78	7.0485	7.5995	7.3000	7.8174	0.2995	0
79	6.7939	7.3832	7.0113	8.6725	0.3718	0
80	6.5301	7.1539	6.7351	9.5526	0.4188	0
81	6.2585	6.9142	6.4079	10.4769	0.5063	0
82	5.9809	6.6658	6.1584	11.4511	0.5074	0
83	5.6993	6.4071	5.8415	12.4189	0.5656	0
84	5.4157	6.1419	5.5351	13.4084	0.6068	0
85	5.0707	5.7396	5.1972	13.1913	0.5424	0
86	4.7335	5.3425	4.7748	12.8645	0.5677	0
87	4.4033	4.9516	4.4311	12.4527	0.5205	0
88	4.0784	4.5650	4.0895	11.9292	0.4754	0
89	3.7567	4.1795	3.8028	11.2541	0.3767	0
90	3.4344	3.7927	3.4631	10.4314	0.3296	0
91	3.1063	3.3998	3.1181	9.4494	0.2816	0
92	2.7639	2.9928	2.7851	8.2828	0.2077	0
93	2.3944	2.5601	2.4024	6.9202	0.1577	0
94	1.9777	2.0836	1.9787	5.3543	0.1049	0
95	1.4815	1.5347	1.4919	3.5940	0.0428	0

United Kingdom

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9295	9.9639	9.9201	0.3459	0.0438	0
51	9.9012	9.9392	9.8976	0.3842	0.0416	0
52	9.8697	9.9121	9.8747	0.4289	0.0374	0
53	9.8350	9.8821	9.8426	0.4794	0.0395	0
54	9.7965	9.8492	9.8147	0.5381	0.0344	0
55	9.7539	9.8129	9.7836	0.6049	0.0293	0
56	9.7069	9.7731	9.7344	0.6822	0.0387	0
57	9.6550	9.7292	9.6814	0.7685	0.0478	0
58	9.5978	9.6811	9.6374	0.8678	0.0436	0
59	9.5348	9.6284	9.5830	0.9821	0.0454	0
60	9.4655	9.5709	9.5279	1.1144	0.0431	0
61	9.3893	9.5078	9.4571	1.2620	0.0507	0
62	9.3058	9.4388	9.3933	1.4293	0.0456	0
63	9.2144	9.3636	9.3217	1.6196	0.0419	0
64	9.1145	9.2815	9.2490	1.8328	0.0325	0
65	9.0055	9.1924	9.1546	2.0754	0.0378	0
66	8.8869	9.0956	9.0497	2.3479	0.0459	0
67	8.7582	8.9905	8.9448	2.6525	0.0457	0
68	8.6189	8.8764	8.8258	2.9886	0.0506	0
69	8.4685	8.7536	8.6810	3.3664	0.0726	0
70	8.3067	8.6215	8.5145	3.7898	0.1070	0
71	8.1333	8.4790	8.3569	4.2503	0.1220	0
72	7.9481	8.3267	8.1704	4.7638	0.1563	0
73	7.7511	8.1641	7.9836	5.3284	0.1805	0
74	7.5426	7.9909	7.7734	5.9440	0.2175	0
75	7.3229	7.8072	7.5462	6.6136	0.2609	0
76	7.0925	7.6122	7.3473	7.3265	0.2649	0
77	6.8524	7.4069	7.1042	8.0913	0.3027	0
78	6.6035	7.1913	6.8808	8.9017	0.3105	0
79	6.3470	6.9658	6.5866	9.7492	0.3792	0
80	6.0844	6.7312	6.3829	10.6301	0.3484	0
81	5.8173	6.4883	5.9044	11.5336	0.5839	0
82	5.5474	6.2379	5.8909	12.4462	0.3469	0
83	5.2765	5.9824	5.5380	13.3776	0.4444	0
84	5.0066	5.7213	5.1813	14.2762	0.5400	0
85	4.6936	5.3517	4.8808	14.0208	0.4710	0
86	4.3889	4.9898	4.5390	13.6915	0.4508	0
87	4.0915	4.6331	4.2364	13.2379	0.3967	0
88	3.7998	4.2807	3.9169	12.6542	0.3638	0
89	3.5116	3.9307	3.6123	11.9336	0.3184	0
90	3.2234	3.5801	3.3223	11.0678	0.2578	0
91	2.9297	3.2226	2.9890	9.9986	0.2336	0
92	2.6223	2.8514	2.6874	8.7364	0.1640	0
93	2.2882	2.4542	2.3306	7.2557	0.1235	0
94	1.9065	2.0128	1.9462	5.5727	0.0665	0
95	1.4433	1.4970	1.4661	3.7164	0.0309	0

Italy

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9952	10.0300	9.9729	0.3482	0.0571	0
51	9.9729	10.0115	9.9500	0.3875	0.0615	0
52	9.9479	9.9909	9.9312	0.4321	0.0597	0
53	9.9202	9.9683	9.9010	0.4854	0.0673	0
54	9.8892	9.9431	9.8723	0.5449	0.0708	0
55	9.8548	9.9153	9.8417	0.6142	0.0736	0
56	9.8164	9.8845	9.8096	0.6933	0.0749	0
57	9.7738	9.8506	9.7678	0.7855	0.0827	0
58	9.7264	9.8129	9.7380	0.8888	0.0749	0
59	9.6739	9.7715	9.6961	1.0094	0.0754	0
60	9.6156	9.7259	9.6642	1.1466	0.0617	0
61	9.5512	9.6757	9.6128	1.3039	0.0629	0
62	9.4798	9.6206	9.5561	1.4846	0.0645	0
63	9.4011	9.5601	9.4888	1.6916	0.0713	0
64	9.3143	9.4936	9.4250	1.9243	0.0685	0
65	9.2189	9.4214	9.3367	2.1963	0.0846	0
66	9.1141	9.3417	9.2515	2.4971	0.0901	0
67	8.9993	9.2550	9.1424	2.8413	0.1125	0
68	8.8738	9.1605	9.0484	3.2306	0.1121	0
69	8.7372	9.0576	8.9299	3.6677	0.1278	0
70	8.5887	8.9462	8.7963	4.1624	0.1499	0
71	8.4281	8.8250	8.6425	4.7093	0.1825	0
72	8.2548	8.6939	8.4846	5.3198	0.2093	0
73	8.0686	8.5530	8.2908	6.0029	0.2622	0
74	7.8695	8.4003	8.1062	6.7442	0.2941	0
75	7.6577	8.2361	7.9058	7.5533	0.3303	0
76	7.4333	8.0614	7.6726	8.4494	0.3887	0
77	7.1970	7.8755	7.4345	9.4279	0.4410	0
78	6.9496	7.6765	7.2051	10.4595	0.4714	0
79	6.6923	7.4678	6.9324	11.5880	0.5354	0
80	6.4263	7.2469	6.6654	12.7691	0.5814	0
81	6.1532	7.0163	6.3386	14.0265	0.6777	0
82	5.8749	6.7746	6.0542	15.3142	0.7204	0
83	5.5933	6.5232	5.7726	16.6261	0.7507	0
84	5.3105	6.2648	5.4799	17.9709	0.7848	0
85	4.9719	5.8485	5.1253	17.6321	0.7232	0
86	4.6413	5.4380	4.7808	17.1667	0.6572	0
87	4.3179	5.0338	4.4134	16.5808	0.6204	0
88	4.0003	4.6335	4.0915	15.8290	0.5421	0
89	3.6862	4.2354	3.7350	14.8984	0.5004	0
90	3.3721	3.8364	3.4317	13.7665	0.4047	0
91	3.0527	3.4319	3.0758	12.4227	0.3561	0
92	2.7197	3.0146	2.7524	10.8440	0.2621	0
93	2.3602	2.5729	2.3592	9.0092	0.2136	0
94	1.9541	2.0896	1.9685	6.9333	0.1211	0
95	1.4685	1.5369	1.4779	4.6569	0.0590	0

France

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9732	10.0057	9.8781	0.3259	0.1277	0
51	9.9499	9.9859	9.8599	0.3625	0.1260	0
52	9.9240	9.9640	9.8402	0.4033	0.1238	0
53	9.8953	9.9399	9.8187	0.4508	0.1212	0
54	9.8636	9.9134	9.7931	0.5052	0.1204	0
55	9.8285	9.8841	9.7676	0.5660	0.1165	0
56	9.7897	9.8520	9.7375	0.6367	0.1145	0
57	9.7469	9.8167	9.7088	0.7165	0.1079	0
58	9.6996	9.7780	9.6780	0.8084	0.1000	0
59	9.6475	9.7356	9.6536	0.9134	0.0820	0
60	9.5901	9.6892	9.6088	1.0330	0.0804	0
61	9.5269	9.6383	9.5763	1.1689	0.0620	0
62	9.4576	9.5825	9.5219	1.3210	0.0606	0
63	9.3814	9.5216	9.4744	1.4942	0.0472	0
64	9.2980	9.4552	9.4181	1.6910	0.0371	0
65	9.2067	9.3828	9.3506	1.9124	0.0321	0
66	9.1071	9.3036	9.2797	2.1583	0.0239	0
67	8.9985	9.2176	9.1884	2.4349	0.0292	0
68	8.8804	9.1243	9.1006	2.7465	0.0237	0
69	8.7524	9.0228	9.0024	3.0894	0.0203	0
70	8.6139	8.9136	8.8893	3.4789	0.0243	0
71	8.4645	8.7953	8.7572	3.9077	0.0381	0
72	8.3040	8.6683	8.6228	4.3877	0.0456	0
73	8.1319	8.5309	8.4586	4.9068	0.0723	0
74	7.9483	8.3842	8.2789	5.4835	0.1053	0
75	7.7531	8.2268	8.0958	6.1102	0.1310	0
76	7.5465	8.0597	7.8918	6.7995	0.1679	0
77	7.3289	7.8818	7.6560	7.5435	0.2257	0
78	7.1008	7.6931	7.4276	8.3408	0.2655	0
79	6.8630	7.4935	7.1598	9.1872	0.3338	0
80	6.6165	7.2834	6.8742	10.0799	0.4092	0
81	6.3624	7.0640	6.5666	11.0273	0.4975	0
82	6.1022	6.8358	6.2816	12.0210	0.5542	0
83	5.8374	6.5983	5.9993	13.0344	0.5990	0
84	5.5697	6.3532	5.6693	14.0682	0.6839	0
85	5.2258	5.9456	5.2436	13.7731	0.7020	0
86	4.8879	5.5413	4.9162	13.3658	0.6250	0
87	4.5549	5.1408	4.5486	12.8635	0.5922	0
88	4.2250	4.7430	4.1859	12.2596	0.5571	0
89	3.8957	4.3436	3.8349	11.4965	0.5087	0
90	3.5631	3.9402	3.5028	10.5823	0.4374	0
91	3.2216	3.5282	3.1149	9.5174	0.4133	0
92	2.8625	3.0994	2.7836	8.2760	0.3158	0
93	2.4731	2.6420	2.3983	6.8302	0.2438	0
94	2.0336	2.1398	1.9928	5.2226	0.1470	0
95	1.5132	1.5658	1.4912	3.4777	0.0746	0

Spain

x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	10.0021	10.0441	9.9400	0.4198	0.1041	0
51	9.9803	10.0275	9.9207	0.4732	0.1068	0
52	9.9559	10.0091	9.8990	0.5346	0.1101	0
53	9.9286	9.9886	9.8751	0.6037	0.1135	0
54	9.8983	9.9658	9.8523	0.6824	0.1135	0
55	9.8644	9.9405	9.8218	0.7710	0.1187	0
56	9.8267	9.9122	9.7985	0.8705	0.1138	0
57	9.7847	9.8809	9.7621	0.9835	0.1188	0
58	9.7380	9.8462	9.7299	1.1115	0.1163	0
59	9.6861	9.8079	9.6928	1.2575	0.1151	0
60	9.6285	9.7653	9.6509	1.4204	0.1144	0
61	9.5646	9.7181	9.6028	1.6044	0.1153	0
62	9.4939	9.6660	9.5457	1.8125	0.1203	0
63	9.4158	9.6084	9.4750	2.0458	0.1334	0
64	9.3295	9.5448	9.4182	2.3074	0.1266	0
65	9.2346	9.4746	9.3565	2.5994	0.1181	0
66	9.1302	9.3973	9.2639	2.9260	0.1334	0
67	9.0157	9.3123	9.1758	3.2901	0.1365	0
68	8.8905	9.2192	9.0714	3.6970	0.1477	0
69	8.7539	9.1179	8.9503	4.1577	0.1676	0
70	8.6055	9.0067	8.8187	4.6625	0.1880	0
71	8.4446	8.8851	8.6776	5.2169	0.2076	0
72	8.2710	8.7529	8.5102	5.8272	0.2428	0
73	8.0843	8.6099	8.3291	6.5018	0.2808	0
74	7.8845	8.4550	8.1320	7.2356	0.3231	0
75	7.6718	8.2884	7.9207	8.0384	0.3678	0
76	7.4463	8.1099	7.7051	8.9111	0.4048	0
77	7.2088	7.9184	7.4577	9.8429	0.4607	0
78	6.9601	7.7134	7.2083	10.8232	0.5052	0
79	6.7013	7.4965	6.9338	11.8661	0.5627	0
80	6.4337	7.2676	6.6326	12.9613	0.6350	0
81	6.1591	7.0268	6.3381	14.0887	0.6886	0
82	5.8791	6.7754	6.0094	15.2453	0.7660	0
83	5.5960	6.5145	5.7310	16.4135	0.7834	0
84	5.3117	6.2439	5.4276	17.5512	0.8163	0
85	4.9730	5.8253	5.0418	17.1385	0.7835	0
86	4.6425	5.4149	4.7064	16.6379	0.7085	0
87	4.3192	5.0093	4.3602	15.9764	0.6491	0
88	4.0019	4.6095	4.0519	15.1848	0.5577	0
89	3.6880	4.2121	3.7089	14.2112	0.5032	0
90	3.3741	3.8155	3.3755	13.0810	0.4400	0
91	3.0548	3.4137	3.0591	11.7495	0.3546	0
92	2.7218	2.9996	2.7038	10.2063	0.2958	0
93	2.3622	2.5614	2.3781	8.4322	0.1833	0
94	1.9557	2.0818	1.9824	6.4498	0.0994	0
95	1.4694	1.5324	1.4751	4.2879	0.0573	0

CBD model: Nonlinear variant

	New Zealand						Sweden						Germany						Finland					
x	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$	$L_x^{\text{mean}}(N)$	$L_x^{\text{Upper}}(N)$	$L_x^*(N)$	CapR	ΔR	${}_j\mathcal{I}_p^x(N)$
50	9.9539	9.9881	9.9513	0.3427	0.0368	0	10.0022	10.0213	9.9639	0.1913	0.0574	0	9.9584	9.9832	9.8861	0.2489	0.0972	0	9.2922	9.5389	9.2446	2.6553	0.2943	0
51	9.9286	9.9658	9.9191	0.3741	0.0466	0	9.9800	10.0010	9.9453	0.2111	0.0558	0	9.9329	9.9598	9.8616	0.2707	0.0982	0	9.2249	9.4839	9.1729	2.8073	0.3109	0
52	9.9006	9.9411	9.9042	0.4089	0.0369	0	9.9551	9.9784	9.9227	0.2336	0.0556	0	9.9046	9.9337	9.8363	0.2942	0.0975	0	9.1530	9.4247	9.0677	2.9677	0.3570	0
53	9.8696	9.9138	9.8706	0.4486	0.0432	0	9.9274	9.9531	9.8928	0.2589	0.0602	0	9.8732	9.9048	9.8068	0.3206	0.0980	0	9.0763	9.3611	9.1476	3.1375	0.2135	0
54	9.8353	9.8840	9.8562	0.4951	0.0278	0	9.8963	9.9249	9.8679	0.2885	0.0569	0	9.8384	9.8729	9.7731	0.3507	0.0998	0	8.9946	9.2934	8.9133	3.3219	0.3801	0
55	9.7974	9.8511	9.8333	0.5478	0.0178	0	9.8616	9.8934	9.8361	0.3222	0.0574	0	9.7998	9.8372	9.7488	0.3822	0.0885	0	8.9075	9.2206	9.0015	3.5144	0.2191	0
56	9.7556	9.8148	9.7846	0.6070	0.0302	0	9.8230	9.8584	9.7988	0.3608	0.0596	0	9.7571	9.7978	9.6991	0.4178	0.0987	0	8.8150	9.1425	8.8733	3.7147	0.2692	0
57	9.7094	9.7750	9.7223	0.6753	0.0527	0	9.7798	9.8195	9.7514	0.4057	0.0681	0	9.7098	9.7541	9.6588	0.4558	0.0953	0	8.7168	9.0591	8.8001	3.9269	0.2590	0
58	9.6585	9.7314	9.6985	0.7541	0.0328	0	9.7317	9.7762	9.7172	0.4567	0.0590	0	9.6577	9.7059	9.6370	0.4992	0.0689	0	8.6127	8.9698	8.8848	4.1460	0.0850	0
59	9.6025	9.6839	9.6489	0.8479	0.0351	0	9.6783	9.7283	9.6675	0.5171	0.0608	0	9.6001	9.6526	9.5802	0.5472	0.0723	0	8.5025	8.8754	8.5613	4.3855	0.3140	0
60	9.5409	9.6318	9.5913	0.9533	0.0405	0	9.6188	9.6752	9.6264	0.5863	0.0488	0	9.5365	9.5940	9.5597	0.6022	0.0343	0	8.3860	8.7743	8.6315	4.6297	0.1428	0
61	9.4732	9.5750	9.5364	1.0754	0.0386	0	9.5528	9.6165	9.5637	0.6674	0.0529	0	9.4666	9.5292	9.4789	0.6617	0.0503	0	8.2632	8.6672	8.5191	4.8893	0.1481	0
62	9.3989	9.5132	9.4576	1.2169	0.0556	0	9.4796	9.5517	9.5224	0.7609	0.0293	0	9.3897	9.4581	9.4277	0.7288	0.0304	0	8.1339	8.5527	8.3686	5.1480	0.1840	0
63	9.3175	9.4457	9.4041	1.3758	0.0415	0	9.3986	9.4800	9.4527	0.8661	0.0273	0	9.3052	9.3803	9.3652	0.8070	0.0151	0	7.9981	8.4323	8.3787	5.4291	0.0536	0
64	9.2284	9.3724	9.3320	1.5602	0.0404	0	9.3091	9.4013	9.3851	0.9902	0.0162	0	9.2126	9.2951	9.2773	0.8954	0.0178	0	7.8557	8.3064	8.1978	5.7367	0.1086	0
65	9.1312	9.2929	9.2379	1.7717	0.0550	0	9.2104	9.3148	9.2898	1.1332	0.0250	0	9.1113	9.2022	9.1905	0.9968	0.0116	0	7.7068	8.1737	8.0594	6.0590	0.1144	0
66	9.0252	9.2062	9.1430	2.0052	0.0632	0	9.1018	9.2196	9.1905	1.2940	0.0291	0	9.0007	9.1008	9.0496	1.1115	0.0512	0	7.5514	8.0349	7.9169	6.4021	0.1180	0
67	8.9100	9.1124	9.0443	2.2721	0.0681	0	8.9827	9.1153	9.1074	1.4768	0.0080	0	8.8803	8.9903	8.9766	1.2389	0.0137	0	7.3897	7.8901	7.8550	6.7714	0.0352	0
68	8.7849	9.0110	8.9082	2.5733	0.1028	0	8.8522	9.0013	8.9778	1.6843	0.0235	0	8.7493	8.8706	8.8504	1.3854	0.0202	0	7.2219	7.7386	7.6654	7.1541	0.0732	0
69	8.6497	8.9016	8.8277	2.9125	0.0739	0	8.7098	8.8768	8.8479	1.9170	0.0289	0	8.6075	8.7409	8.7027	1.5499	0.0382	0	7.0483	7.5816	7.4584	7.5662	0.1232	0
70	8.5037	8.7835	8.6591	3.2907	0.1245	0	8.5549	8.7414	8.7466	2.1797	-0.0052	1	8.4542	8.6003	8.5501	1.7275	0.0502	0	6.8691	7.4178	7.3445	7.9876	0.0733	0
71	8.3467	8.6568	8.5435	3.7148	0.1132	0	8.3871	8.5952	8.5814	2.4813	0.0137	0	8.2892	8.4490	8.4011	1.9278	0.0478	0	6.6848	7.2489	7.1416	8.4373	0.1072	0
72	8.1784	8.5205	8.3845	4.1835	0.1360	0	8.2058	8.4366	8.4248	2.8128	0.0119	0	8.1121	8.2864	8.2249	2.1487	0.0615	0	6.4959	7.0741	6.9569	8.9003	0.1172	0
73	7.9986	8.3747	8.1917	4.7022	0.1831	0	8.0110	8.2660	8.2555	3.1833	0.0105	0	7.9228	8.1123	8.0629	2.3917	0.0493	0	6.3028	6.8939	6.7232	9.3776	0.1707	0
74	7.8074	8.2193	8.0180	5.2767	0.2014	0	7.8026	8.0823	8.0369	3.5845	0.0454	0	7.7214	7.9266	7.8924	2.6579	0.0342	0	6.1063	6.7103	6.5745	9.8922	0.1358	0
75	7.6048	8.0540	7.7942	5.9076	0.2598	0	7.5807	7.8865	7.8595	4.0342	0.0270	0	7.5079	7.7290	7.6943	2.9449	0.0348	0	5.9068	6.5218	6.3831	10.4102	0.1387	0
76	7.3911	7.8779	7.5782	6.5856	0.2997	0	7.3459	7.6779	7.5841	4.5206	0.0939	0	7.2829	7.5206	7.4687	3.2639	0.0519	0	5.7052	6.3320	6.1887	10.9861	0.1433	0
77	7.1669	7.6925	7.3687	7.3329	0.3238	0	7.0987	7.4571	7.3422	5.0481	0.1149	0	7.0470	7.3007	7.2441	3.6007	0.0566	0	5.5022	6.1390	5.8660	11.5725	0.2729	0
78	6.9329	7.4958	7.1080	8.1200	0.3878	0	6.8403	7.2235	7.0660	5.6024	0.1575	0	6.8009	7.0705	6.9665	3.9649	0.1041	0	5.2985	5.9424	5.8095	12.1510	0.1328	0
79	6.6899	7.2900	6.7526	8.9701	0.5374	0	6.5718	6.9782	6.7528	6.1833	0.2253	0	6.5457	6.8299	6.6449	4.3408	0.1850	0	5.0950	5.7438	5.4582	12.7336	0.2856	0
80	6.4392	7.0744	6.5558	9.8643	0.5186	0	6.2949	6.7228	6.4898	6.7971	0.2330	0	6.2829	6.5807	6.3747	4.7390	0.2060	0	4.8924	5.5433	5.1802	13.3060	0.3632	0
81	6.1819	6.8502	6.1917	10.8102	0.6586	0	6.0114	6.4577	6.1743	7.4239	0.2834	0	6.0139	6.3231	5.8474	5.1423	0.4757	0	4.6914	5.3428	4.8733	13.8842	0.4695	0
82	5.9197	6.6169	5.9698	11.7768	0.6471	0	5.7233	6.1851	5.8939	8.0685	0.2912	0	5.7404	6.0592	6.0839	5.5538	-0.0247	1	4.4929	5.1421	4.6195	14.4498	0.5227	0
83	5.6541	6.3773	5.6619	12.7912	0.7154	0	5.4328	5.9060	5.5279	8.7115	0.3782	0	5.4643	5.7906	5.5173	5.9727	0.2734	0	4.2976	4.9429	4.5215	15.0166	0.4214	0
84	5.3868	6.1305	5.3041	13.8045	0.8264	0	5.1421	5.6226	5.2461	9.3443	0.3765	0	5.1875	5.5186	5.1217	6.3831	0.3969	0	4.1060	4.7445	4.2097	15.5508	0.5348	0
85	5.0557	5.7415	5.0171	13.5653	0.7244	0	4.8057	5.2492	4.8590	9.2282	0.3902	0	4.8542	5.1566	4.9300	6.2301	0.2266	0	3.8972	4.4884	3.9581	15.1713	0.5303	0
86	4.7310	5.3581	4.5849	13.2555	0.7732	0	4.4788	4.8844	4.4997	9.0578	0.3847	0	4.5299	4.8036	4.3553	6.0425	0.4483	0	3.6925	4.2351	3.6299	14.6938	0.6051	0
87	4.4115	4.9767	4.2225	12.8116	0.7542	0	4.1607	4.5278	4.1952	8.8243	0.3326	0	4.2136	4.4587	4.1629	5.8150	0.2958	0	3.4909	3.9839	3.4169	14.1235	0.5670	0
88	4.0958	4.5975	4.0022	12.2495	0.5952	0	3.8502	4.1775	3.8725	8.5029	0.3050	0	3.9041	4.1208	3.8505	5.5500	0.2702	0	3.2905	3.7328	3.2340	13.4417	0.4988	0
89	3.7811	4.2183	3.6693	11.5621	0.5489	0	3.5452	3.8316	3.5470	8.0795	0.2846	0	3.5989	3.7868	3.5331	5.2227	0.2537	0	3.0888	3.4780	2.9343	12.5990	0.5437	0
90	3.4637	3.8343	3.3595	10.6998	0.4748	0	3.2425	3.4871	3.2705	7.5433	0.2165	0	3.2945	3.4532	3.2417	4.8148	0.2115	0	2.8820	3.2168	2.7585	11.6171	0.4583	0
91	3.1379	3.4411	3.0227	9.6643	0.4185	0	2.9369	3.1387	2.9281	6.8701	0.2106	0	2.9857	3.1153	2.9286	4.3418	0.1868	0	2.6643	2.9424	2.5201	10.4376	0.4223	0
92	2.7951	3.0313	2.6454	8.4497	0.3859	0	2.6206	2.7792	2.6073	6.0502	0.1719	0	2.6642	2.7653	2.6256	3.7964	0.1397	0	2.4267	2.6476	2.2914	9.1034	0.3562	0
93	2.4224	2.5930	2.3352	7.0443	0.2579	0	2.2809	2.3968	2.2668	5.0809	0.1300	0	2.3172	2.3907	2.2867	3.1735	0.1040	0	2.1552	2.3172	2.0314	7.5198	0.2858	0
94	1.9996	2.1084	1.9375	5.4410	0.1709	0	1.8974	1.9726	1.9003	3.9634	0.0722	0	1.9242	1.9715	1.9028	2.4544	0.0686	0	1.8273	1.9327	1.7411	5.7676	0.1916	0
95	1.4950	1.5499	1.4813	3.6740	0.0686	0	1.4359	1.4746	1.4420	2.6956	0.0326	0	1.4518	1.4757	1.4466	1.6456	0.0292	0	1.4067	1.4609	1.3702	3.8583	0.0908	0

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